Evolutionary and Spectral Problems Generated by a Problem on Small Movements of Visco-elastic Fluid

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Introduction

In this article we investigate a class of integrodifferential operator equations in the Hilbert space. These equations are generated by initial boundary value problem and spectral problem on small movements of viscoelastic fluid in the completely filled container.

One of models of such fluids is Oldroid's model. It is described, for example, in the book [1]. We use a generalized model that goes into Oldroid's model for $m = 1$ (see (1.5)).

This model was considered by A. I. Miloslavskii in [2]–[6]. It should be noted that the appearance of the present article is significantly stipulated by his works. An abstract integrodifferential equation (1.1) with unbounded operator coefficients A_k $(k = \overline{0, m})$ is input equation for the investigation. The existence theorem for strong solution of this equation is established with the use of the theory of contracting semigroups. The spectral problem (see (2.53)) corresponding to equation (1.1) is investigated in detail. The present paper is based on the works [7]–[12].

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1 Existence theorem for a strong solution of an evolutionary problem

1.1 Setting of a problem

Let us consider the following Cauchy problem for integro-differential equation

$$\frac{du}{dt} + A_0 u + \sum_{k=1}^{m} \int\limits_0^t e^{-\gamma_k (t-s)} A_k u(s)\, ds = f(t), \quad u(0) = u^0. \tag{1.1}$$

in a separable Hilbert space $\mathcal{H}$. Here $u = u(t)$ is the unknown function with values in $\mathcal{H}$, γ_k $(k = \overline{0,m})$ are positive constants: $0 < \gamma_1 < \ldots < \gamma_m < \infty$, $f(t)$ is the given function with values in $\mathcal{H}$, $u^0 \in \mathcal{H}$. By A_k $(k = \overline{0,m})$ in (1.1) we denote unbounded selfadjoint positive definite operators ($A_k \gg 0$, $k = \overline{0,m}$) defined on $\mathcal{D}(A_k)$ $(k = \overline{0,m})$ such that

$$\mathcal{D}(A_k) = \mathcal{D}(A_0) \quad (\, k = \overline{1,m}\,), \quad 0 < A_k^{-1} \in \mathfrak{S}_\infty \qquad (\, k = \overline{0,m}\,). \tag{1.2}$$

In the Hilbert space $\mathcal{H} = L_2(\Omega)$ $(\Omega \subset \mathbb{R}^{\ltimes})$ we can consider as an example of such operators a set of uniformly elliptic operators

$$A_k u := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}^{(k)}(x) \frac{\partial u}{\partial x_j} \right) \qquad (k = \overline{0,m}), \tag{1.3}$$

defined on the same domain

$$\mathcal{D}(A_k) = H_0^2(\Omega) := \{\, u(x) \in H^2(\Omega) : \quad u = 0 \quad (\ \partial\Omega\,)\,\}, \tag{1.4}$$

where $H^2(\Omega)$ is the standard Sobolev's space, $\partial\Omega \in C^2$.

1.2 A hydrodynamic application

A problem on small oscillations of visco-elastic fluid (see for instance [1]) filling an arbitrary domain $\Omega \subset \mathbb{R}^{\not\models}$, is an important special case of problem (1.1).

This problem is to find velocity field $\vec{u}(t,x)$ and pressure field $p(t,x)$ from the following system of equations and boundary and initial conditions.

$$\frac{\partial \vec{u}}{\partial t} = -\rho^{-1} \nabla p + \nu I_0(t)(\Delta \vec{u}) + \vec{f}, \qquad \operatorname{div} \vec{u} = 0 \qquad (\ \Omega\,), \tag{1.5}$$

$$(I_0(t)\vec{v})\,(t,x) := \vec{v}(t,x) + \sum_{j=1}^{m} \alpha_j \int\limits_0^t e^{-\gamma_j (t-s)} \vec{v}(s,x)\, ds,$$

$$\vec{u} = \vec{0} \qquad (\ S = \partial\Omega\,), \qquad \vec{u}(0,x) = \vec{u}^0(x),$$

where $\nu > 0$ and $\rho > 0$ are the coefficient of kinematic viscosity and the density of the fluid respectively, $\alpha_j > 0$ $(j = \overline{1,m})$, $0 < \gamma_1 < \ldots < \gamma_m$, and $\vec{f} = \vec{f}(t,x)$ is the density of a small field of mass forces superimposed on the gravitational field.

To pass from (1.5) to the problem of the form (1.1) we suppose the functions $\vec{u}(t,x)$ and $\nabla p(t,x)$ for all t to be an element of the Hilbert space of vector functions $\vec{L}_2(\Omega)$ with the inner product

$$(\vec{u},\vec{v})_\Omega := \int_\Omega \sum_{k=1}^{3} u_k(x)\overline{v_k(x)}\, d\Omega. \tag{1.6}$$

We shall use an orthogonal decomposition

$$\vec{L}_2(\Omega) = \vec{J}_0(\Omega) \oplus \vec{G}(\Omega), \quad \vec{G}(\Omega) := \{\, \vec{v} \in \vec{L}_2(\Omega) : \vec{v} = \nabla p \,\}, \tag{1.7}$$

$$\vec{J}_0(\Omega) := \{\, \vec{u} \in \vec{L}_2(\Omega) :\ \operatorname{div} \vec{u} = 0 \,(\text{in } \Omega),\ u_n := \vec{u}\cdot\vec{n} = 0\,(\text{ on } \partial\Omega\,)\,\}, \tag{1.8}$$

where $\vec{n}$ is the unit vector of the normal to $\partial\Omega$, $\vec{u}$ u_n are distributions of finite order (see, for instance, [15, §2.1]). Taking into account (1.5), we have $\vec{u}(t,x) \in \vec{J}_0(\Omega)$, $\nabla p(t,x) \in \vec{G}(\Omega)$.

Let P_0 be an orthoprojector onto $\vec{J}_0(\Omega)$, and P_G – onto $\vec{G}(\Omega)$, $P_0 + P_G = I$. We suppose the functions $\vec{u}(t,x)$ and $\nabla p(t,x)$ are the classical solution of problem (1.5). Applying the orthoprojectors P_0 and P_G to both sides of the first equation of (1.5), we get

$$\frac{d\vec{u}}{dt} + \nu I_0(t)(A_0\vec{u}) = \vec{f}_0(t), \quad \vec{u}(0) = \vec{u}^0, \tag{1.9}$$

$$A_0\vec{u} := -P_0(\Delta\vec{u}), \quad \vec{f}_0 := P_0\vec{f}, \tag{1.10}$$

$$\rho^{-1}\nabla p = \nu I_0(t)P_G(\Delta\vec{u}) + P_G\vec{f}. \tag{1.11}$$

It follows from (1.11) that pressure field $\nabla p(t,x)$ is immediately found if we know velocity field $\vec{u}(t,x)$ as a function of t with values in $\vec{H}_0^2(\Omega)\bigcap\vec{J}_0(\Omega)$, where $\vec{H}_0^2(\Omega)$ is a space of vector-functions with components from $H_0^2(\Omega)$. Therefore it is sufficient to consider problem (1.9), where $\vec{u} = \vec{u}(t)$ is a function with values in $\vec{J}_0(\Omega)$, and operator A_0 is Stokes operator frequently arising in hydrodynamic problems (see [16], [17]).

Let the boundary $\partial\Omega$ of the domain Ω be twice continuously differentiable $(\partial\Omega \in C^2\,)$; then (see [17])

$$\mathcal{D}(A_0) = \vec{J}_0^2(\Omega) := \{\vec{u} \in \vec{J}_0(\Omega) : \quad \Delta\vec{u} \in \vec{L}_2(\Omega), \quad \vec{u} = \vec{0}\ (\ \partial\Omega\,)\,\}. \tag{1.12}$$

Note also that the operator A_0 is positive definite in $\vec{J}_0(\Omega)$, and $0 < A_0^{-1} \in \mathfrak{S}_\infty$. Eigenvalues $\{\lambda_j(A_0)\}_{j=1}^\infty$ of the operator A_0 form a discrete spectrum and have the following asymptotic behavior [18]

$$\lambda_j(A_0) = \left(\frac{\operatorname{mes}\Omega}{3\pi^2}\right)^{-2/3} j^{2/3}[1 + o(1)] \quad (j \to \infty). \tag{1.13}$$

The generalized Oldroid model of the visco-elastic fluid [2]–[6] corresponds to problem (1.9) for $m \geq 2$. For $m = 1$ we get the Oldroid model (see [1]). Obviously, equation (1.9) is a special case of problem (1.1) in the space $\mathcal{H} = \vec{J}_0(\Omega)$; in this case A_0 should be replaced by νA_0, and A_k – by $\nu\alpha_k A_0$ $(k = \overline{1,m})$, where A_0 is Stokes operator (1.10), (1.12).

1.3 Passage to a different equation in an orthogonal sum of spaces

The form of problem(1.1) and properties of the operator coefficients A_k allow to pass from the integro-differential equation in the Hilbert space $\mathcal{H}$ to a differential equation of the first order in the space $\widetilde{\mathcal{H}}$, which is the orthogonal sum of $m+1$ copies of the space $\mathcal{H}$.

Definition 1.1 *A function $u : [0,T] \to \mathcal{H}$ is said to be a strong solution of problem* (1.1) *on the segment* $[0,T]$ *if the following conditions hold:*

) *u is a strong continuously differentiable in $\mathcal{H}$ function:* *$u(t) \in C^1[0,T;\mathcal{H}]$;*

) *$u(t) \in \mathcal{D}(A_0) = \mathcal{D}(A_k)$, $k = \overline{1,m}$, and $A_k u(t) \in C[0,T;\mathcal{H}]$, $k = \overline{0,m}$, for all $t \in [0,T]$;*

) *u turns equation* (1.1) *into the identity, $u(0) = u^0$.*

Let $u(t)$ be a strong solution of problem (1.1). Let us introduce new unknown functions $u_k(t)$ $(k = \overline{0,m})$ according to the formulas

$$u_0(t) := u(t), \quad u_k(t) := \int_0^t e^{-\gamma_k(t-s)} A_k^{1/2} u_0(s)\, ds, \quad k = \overline{1,m}. \tag{1.14}$$

As $u(t)$ is a continuously differentiable in t function and for all $t \in [0,T]$ belongs to $\mathcal{D}(A_0) = \mathcal{D}(A_k)$, functions $u_k(t)$ are continuously differentiable and

$$\begin{gathered} \frac{du_k}{dt} = \frac{d}{dt}\int_0^t e^{-\gamma_k(t-s)} A_k^{1/2} u_0(s)\, ds = A_k^{1/2} u_0(t) - \\ \gamma_k \int_0^t e^{-\gamma_k(t-s)} A_k^{1/2} u_0(s)\, ds = A_k^{1/2} u_0(t) - \gamma_k u_k(t), \quad k = \overline{1,m}. \end{gathered} \tag{1.15}$$

Equations (1.15) together with (1.1), (1.14) lead to the differential equation

$$\frac{d\tilde{u}}{dt} + \mathcal{A}_0\tilde{u} = \tilde{f}(t), \qquad \tilde{u}(0) = \tilde{u}^0, \tag{1.16}$$

in the Hilbert space

$$\widetilde{\mathcal{H}} := \mathcal{H}_0 \oplus \widehat{\mathcal{H}}_1, \quad \mathcal{H}_0 := \mathcal{H}, \quad \widehat{\mathcal{H}}_1 := \bigoplus_{k=1}^{m} \mathcal{H}_k, \quad \mathcal{H}_k := \mathcal{H}\ (\,k = \overline{1,m}\,), \tag{1.17}$$

where

$$\begin{gathered} \tilde{u}(t) := (u_0(t); \hat{u}_1(t))^t, \quad \hat{u}_1(t) := (u_1(t); \ldots; u_m(t))^t, \\ \tilde{f}(t) := (f(t); \hat{0})^t, \quad \tilde{u}(0) = (u^0; \hat{0})^t. \end{gathered} \tag{1.18}$$

The operator $\mathcal{A}_0$ has the following matrix representation in the orthogonal decomposition (1.17)

$$\begin{gathered} \mathcal{A}_0 := (A_{ij})_{i,j=0}^{1}\,, \qquad A_{00} := A_0, \quad A_{01} := (A_1^{1/2}; \ldots; A_m^{1/2}), \\ A_{10} := -(A_1^{1/2}; \ldots; A_m^{1/2})^t, \qquad A_{11} := \operatorname{diag}(\gamma_k I)_{k=1}^{m}. \end{gathered} \tag{1.19}$$

1.4 Properties of the operator coefficient of the differential equation

The further investigation of the properties of problem (1.1) and problem (1.16) associated with it, is based on studying of properties of the matrix operator $\mathcal{A}_0$.

First note that it follows from $A_k \gg 0$ ($k = \overline{0,m}$) and (1.2) that

$$\mathcal{D}(A_k^\alpha) = \mathcal{D}(A_0^\alpha), \quad 0 \le \alpha \le 1, \quad k = \overline{1,m}, \tag{1.20}$$

by the known Heinz inequality (see [19, .254]). in addition, there are positive constants $c_{k,\alpha}$ and $d_{k,\alpha}$ such that

$$c_{k,\alpha} \le \|A_k^\alpha u\| / \|A_0^\alpha u\| \le d_{k,\alpha} \quad (\,0 \le \alpha \le 1,\ k = \overline{1,m}\,). \tag{1.21}$$

Lemma 1.2 *Suppose the operator $\mathcal{A}_0$ is defined by formulas* (1.19) *on the set*

$$\begin{gathered} \mathcal{D}(\mathcal{A}_0) := \mathcal{D}(A_0) \oplus \widehat{\mathcal{D}}_1^{1/2}, \\ \widehat{\mathcal{D}}_1^{1/2} := \bigoplus_{k=1}^{m} \mathcal{D}(A_k^{1/2}), \quad \mathcal{D}(A_k^{1/2}) = \mathcal{D}(A_0^{1/2})\ (\,k = \overline{1,m}\,), \end{gathered} \tag{1.22}$$

which is dense in $\widetilde{\mathcal{H}}$. Then A_0 is a uniformly accretive operator, i. e;

$$\operatorname{Re}(\mathcal{A}_0\tilde{u}, \tilde{u})_{\widetilde{\mathcal{H}}} \ge c\|\tilde{u}\|^2_{\widetilde{\mathcal{H}}} \qquad (\tilde{u} \in \mathcal{D}(\mathcal{A}_0)), \quad c = \min(\,\lambda_1(A_0); \gamma_1\,) > 0, \tag{1.23}$$

where $\lambda_1(A_0)$ is the minimal eigenvalue of the operator A_0.

Proof. It follows from (1.19) that for $\tilde{u} = (u_0; \hat{u}_1)^t \in \mathcal{D}(\mathcal{A}_0)$

$$\mathrm{Re}\left\{\left(\begin{pmatrix} 0 & A_{01} \\ A_{10} & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ \hat{u}_1 \end{pmatrix}, \begin{pmatrix} u_0 \\ \hat{u}_1 \end{pmatrix}\right)_{\widetilde{\mathcal{H}}}\right\}$$

$$= \mathrm{Re}\left\{\sum_{k=1}^{m}(A_k^{1/2}u_k, u_0)_{\mathcal{H}} - \sum_{k=1}^{m}(A_k^{1/2}u_0, u_k)_{\mathcal{H}}\right\}$$

$$= \mathrm{Re}\sum_{k=1}^{m}\{(A_k^{1/2}u_k, u_0)_{\mathcal{H}} - (u_0, A_k^{1/2}u_k)_{\mathcal{H}}\} = 0.$$

That's why

$$\mathrm{Re}\,(\mathcal{A}_0\tilde{u}, \tilde{u})_{\widetilde{\mathcal{H}}} = \mathrm{Re}\left\{\left(\begin{pmatrix} A_{00} & 0 \\ 0 & A_{11} \end{pmatrix}\begin{pmatrix} u_0 \\ \hat{u}_1 \end{pmatrix}, \begin{pmatrix} u_0 \\ \hat{u}_1 \end{pmatrix}\right)_{\widetilde{\mathcal{H}}}\right\}$$

$$= (A_0u_0, u_0)_{\mathcal{H}} + \sum_{k=1}^{m}\gamma_k\|u_k\|^2_{\mathcal{H}} \geq c\sum_{k=0}^{m}\|u_k\|^2_{\mathcal{H}} = c\|\tilde{u}\|^2_{\widetilde{\mathcal{H}}}, \quad \tilde{u} \in \mathcal{D}(\mathcal{A}_0).$$

■

Lemma 1.3 *The matrix operator $\mathcal{A}_0$ (1.19) is an essentially maximal accretive operator, i. e; its closure $\mathcal{A} = \overline{\mathcal{A}_0}$ is a maximal accretive operator.*

Proof. Since any densely defined accretive operator can be expanded to a closed maximal accretive operator and its adjoint is maximal accretive too [20] (see also [21, p.106–110]), any uniformly accretive operator can be expanded to a maximal accretive operator, which is uniform together with its adjoint. $\mathcal{A}_0$ is uniformly accretive and therefore continuously invertible; then the closure of the operator $\mathcal{A}_0$ is maximal accretive if and only if the closure of its range of values $\mathcal{R}(\mathcal{A}_0)$ coincides with the whole space $\widetilde{\mathcal{H}}$. Let us show that in our conditions $\overline{\mathcal{R}(\mathcal{A}_0)} = \widetilde{\mathcal{H}}$.

Let $\tilde{v} = (v_0; \hat{v}_1)^t \in \widetilde{\mathcal{H}}$ be orthogonal to $\mathcal{R}(\mathcal{A}_0)$. Then, by (1.19), the following relation holds

$$(A_{00}u_0 + A_{01}\hat{u}_1, v_0)_{\mathcal{H}} + (A_{10}u_0 + A_{11}\hat{u}_1, \hat{v}_1)_{\widehat{\mathcal{H}}_1} = (A_0u_0, v_0)_{\mathcal{H}}$$

$$+ \sum_{k=1}^{m}(A_k^{1/2}u_k, v_0)_{\mathcal{H}} - \sum_{k=1}^{m}(A_k^{1/2}u_0, v_k)_{\mathcal{H}} + \sum_{k=1}^{m}\gamma_k(u_k, v_k)_{\mathcal{H}} = 0, \tag{1.24}$$

$$\forall\, u_0 \in \mathcal{D}(\mathcal{A}_0), \qquad \forall\, \hat{u}_1 = (u_1; \ldots; u_m)^t \in \widehat{\mathcal{D}}_1^{1/2}.$$

Suppose here $u_0 = 0$. Then

$$(A_{01}\hat{u}_1, v_0)_{\mathcal{H}} + (A_{11}\hat{u}_1, \hat{v}_1)_{\widehat{\mathcal{H}}_1} = \sum_{k=1}^{m}(A_k^{1/2}u_k, v_0)_{\mathcal{H}} + \sum_{k=1}^{m}\gamma_k(u_k, v_k)_{\mathcal{H}} = 0. \quad (1.25)$$

This formula show that the element $v_0 \in \mathcal{H}_0 = \mathcal{H}$ belongs to the domain of definition of the operator A_{01}^* ($v_0 \in \mathcal{D}(A_{01}^*) = \mathcal{D}(A_k^{1/2}) = \mathcal{D}(A_0^{1/2})$) and

$$A_{01}^* v_0 = -A_{11}\hat{v}_1. \quad (1.26)$$

Assume now in (1.24) $\hat{u}_1 = 0$, $u_0 \in \mathcal{D}(A_0)$. Then taking into account (1.26), we obtain

$$\begin{gathered}(A_{00}u_0, v_0)_{\mathcal{H}} + (A_{10}u_0, \hat{v}_1)_{\widehat{\mathcal{H}}_1} = \\ (A_0^{1/2}u_0, A_0^{1/2}v_0)_{\mathcal{H}} + \sum_{k=1}^{m}\gamma_k^{-1}(A_k^{1/2}u_0, A_k^{1/2}v_0)_{\mathcal{H}} = 0.\end{gathered} \quad (1.27)$$

Let us now introduce in $\mathcal{D}(A_0^{1/2}) = \mathcal{D}(A_k^{1/2})$ ($k = \overline{1,m}$) two new scalar products

$$\begin{gathered}[u_0, v_0] := (A_0^{1/2}u_0, A_0^{1/2}v_0)_{\mathcal{H}}, \quad u_0, v_0 \in \mathcal{D}(A_0^{1/2}), \\ < u_0, v_0 > := [u_0, v_0] + \sum_{k=1}^{m}\gamma_k^{-1}(A_k^{1/2}u_0, A_k^{1/2}v_0)_{\mathcal{H}},\end{gathered} \quad (1.28)$$

and the correspondent norms as well. It is easy to see that these norms are equivalent:

$$1 \leq \frac{< u_0, u_0 >}{[u_0, u_0]} \leq 1 + \sum_{k=1}^{m}\gamma_k^{-1}\|A_k^{1/2}A_0^{-1/2}\|^2. \quad (1.29)$$

It follows from here that $< u_0, v_0 > = [Tu_0, v_0] = [u_0, Tv_0]$, where T is a positive definite bounded operator acting in the energetic space $\mathcal{H}_{A_0} = \mathcal{D}(A_0^{1/2})$ with the scalar product $[u_0, v_0]$.

In the terms of scalar products(1.28) equality (1.27) gives the relation

$$< u_0, v_0 > = [u_0, Tv_0] = 0, \qquad \forall u_0 \in \mathcal{D}(\mathcal{A}_{\prime}). \quad (1.30)$$

Since $\mathcal{D}(A_0)$ is dense in $\mathcal{H}_{A_0}$, from (1.30) we have $Tv_0 = 0$, and then from bounded invertibility of T and A_{11} (see (1.19)) we get $v_0 = 0$, $\hat{v}_1 = 0$, i. e; $\tilde{v} = \tilde{0}$. ■

Let us introduce operators

$$\begin{gathered}Q_k := A_k^{1/2}A_0^{-1/2}, \quad \mathcal{D}(Q_k) = \mathcal{H}_k = \mathcal{H}_{\prime}, \\ Q_k^+ := A_0^{-1/2}A_k^{1/2}, \quad \mathcal{D}(Q_k^+) := \mathcal{D}(A_k^{1/2}), \qquad k = \overline{1,m},\end{gathered} \quad (1.31)$$

and also operator rows and columns

$$Q_{10} := (Q_1; \dots ; Q_m)^t, \qquad \mathcal{D}(Q_{10}) = \mathcal{H}_0, \tag{1.32}$$

$$Q_{01}^{+} := (Q_1^{+}; \dots ; Q_m^{+}), \qquad \mathcal{D}(Q_{01}^{+}) = \widehat{\mathcal{D}}_1^{1/2} \subset \widehat{\mathcal{H}}_1. \tag{1.33}$$

Lemma 1.4 *The following relations are valid*

$$Q_k^{+} = Q_k^{*}|\mathcal{D}(A_k^{1/2}) \quad (k = \overline{1, m}), \qquad Q_{01}^{+} = Q_{10}^{*}|\widehat{\mathcal{D}}_1^{1/2}, \tag{1.34}$$

and the closure on continuity of the operator Q_k^{+} coincides with Q_k^{*}, and operator Q_{01}^{+} – with Q_{10}^{*}.

Proof of the lemma follows immediately from the definition of the adjoint operator, formulas (1.31)–(1.33) and from the fact that the operators Q_k, and therefore operators Q_{10}, are defined od the whole $\mathcal{H}_0$. ■

The following theorem is a corollary of lemmas 1.2–1.4.

Theorem 1.5 *There exist the following factorizations of the operator $\mathcal{A}_0$ from* (1.19)*:*

) in the Schur – Frobenius form

$$\mathcal{A}_0 = \begin{pmatrix} I_0 & 0 \\ -Q_{10}A_0^{-1/2} & \hat{I}_1 \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ 0 & A_{11} + Q_{10}Q_{01}^{+} \end{pmatrix} \begin{pmatrix} I_0 & A_0^{-1/2}Q_{01}^{+} \\ 0 & \hat{I}_1 \end{pmatrix}, \tag{1.35}$$

) with the symmetric bordering

$$\mathcal{A}_0 = \begin{pmatrix} A_0^{1/2} & 0 \\ 0 & \hat{I}_1 \end{pmatrix} \begin{pmatrix} I_0 & Q_{01}^{+} \\ -Q_{10} & A_{11} \end{pmatrix} \begin{pmatrix} A_0^{1/2} & 0 \\ 0 & \hat{I}_1 \end{pmatrix}. \tag{1.36}$$

The closure $\mathcal{A}$ of the operator $\mathcal{A}_0$ can be represented

) in the Schur – Frobenius form

$$\mathcal{A} = \begin{pmatrix} I_0 & 0 \\ -Q_{10}A_0^{-1/2} & \hat{I}_1 \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ 0 & A_{11} + Q_{10}Q_{10}^{*} \end{pmatrix} \begin{pmatrix} I_0 & A_0^{-1/2}Q_{10}^{*} \\ 0 & \hat{I}_1 \end{pmatrix}, \tag{1.37}$$

) with the symmetric bordering

$$\mathcal{A} = \begin{pmatrix} A_0^{1/2} & 0 \\ 0 & \hat{I}_1 \end{pmatrix} \begin{pmatrix} I_0 & Q_{10}^{*} \\ -Q_{10} & A_{11} \end{pmatrix} \begin{pmatrix} A_0^{1/2} & 0 \\ 0 & \hat{I}_1 \end{pmatrix}. \tag{1.38}$$

The operator $\mathcal{A}$ is defined on the domain

$$\mathcal{D}(\mathcal{A}) = \{\, \tilde{u} = (u_0; \hat{u}_1)^t \in \widetilde{\mathcal{H}} : \quad u_0 + A_0^{-1/2} Q_{10}^* \hat{u}_1 \in \mathcal{D}(A_0) \,\} \tag{1.39}$$

by the formula

$$\mathcal{A}\tilde{u} = \begin{pmatrix} A_0(u_0 + A_0^{-1/2} Q_{10}^* \hat{u}_1) \\ \\ -Q_{10} A_0^{1/2} u_0 + A_{11}\hat{u}_1 \end{pmatrix}, \quad \tilde{u} \in \mathcal{D}(\mathcal{A}). \tag{1.40}$$

Proof. Formulas (1.35), (1.36) can be checked immediately on the elements from $\mathcal{D}(\mathcal{A}_0)$. Further, the second and third factors from the right in (1.35) allows the closures by replacing of the operator Q_{01}^+ by Q_{10}^*, and the operator $\mathcal{A}$ from (1.37) appears. After that every factor in (1.37) is a closed operator having a bounded inverse, and that's why the whole product (1.37) is a closed operator. It follows from lemma 1.3 that the operator $\mathcal{A}$ from (1.37) is a maximal uniformly accretive operator such that inequality (1.23) holds.

Similar arguments allow to state that the operator $\mathcal{A}$ from (1.38), which is the expansion of (1.36), is maximal uniformly accretive. Here the extreme factors are unbounded selfadjoint operators having the bounded inverse, and the middle factor is a bounded operator having the bounded inverse

$$\begin{pmatrix} I_0 & Q_{10}^* \\ \\ -Q_{10} & A_{11} \end{pmatrix}^{-1} = \begin{pmatrix} (I_0 + Q_{10}^* A_{11}^{-1} Q_{10})^{-1} & -Q_{10}^*(A_{11} + Q_{10}Q_{10}^*)^{-1} \\ \\ (A_{11} + Q_{10}Q_{10}^*)^{-1} Q_{10} & (A_{11} + Q_{10}Q_{10}^*)^{-1} \end{pmatrix}. \tag{1.41}$$

Thus the operator $\mathcal{A}$ represented in the form (1.37) or (1.38) is maximal uniformly accretive, it is defined on the domain (1.39), and this implies that $u_0 \in \mathcal{D}(A_0^{1/2})$. Then in each of its factorizations (1.37) and (1.38) the result of applying of the matrix-factors to an arbitrary element from $\mathcal{D}(\mathcal{A})$ makes sense, and the result is expressed by formula (1.40) in the both cases. ∎

1.5 Theorems on correct solvability

Properties of the matrix operator $\mathcal{A}$ proved in subsection 1.4, allow to reduce problem (1.1) to the well investigated problem for a differential equation of the first order with a maximal dissipative operator coefficient.

Theorem 1.6 *Suppose that in problem* (1.1) *the following conditions hold:*

$$u_0 \in \mathcal{D}(A_0), \quad f(t) \in C^1[0, T; \mathcal{H}]. \tag{1.42}$$

Then this problem has a unique strong solution on $[0, T]$.

Proof. Consider instead of (1.16) the Cauchy problem

$$\frac{d\tilde{u}}{dt} + \mathcal{A}\tilde{u} = \tilde{f}(t), \qquad \tilde{u}(0) = \tilde{u}^0, \tag{1.43}$$

where $\tilde{u}(t)$, $\tilde{f}(t)$ and $\tilde{u}(0)$ are expressed by formulas (1.18). Then it follows from (1.39) and (1.42) that $\tilde{u}(0) \in \mathcal{D}(\mathcal{A})$, and $\tilde{f}(t) \in C^1[0,T;\widetilde{\mathcal{H}}]$. As $\mathcal{A}$ is maximal accretive operator acting in the Hilbert space $\widetilde{\mathcal{H}}$, problem (1.43) has a unique strong solution (see, for example, [21, p. 166] and also [22]–[25]) on the segment $[0,T]$.

For this solution (1.43) holds, i. e; the function

$$\tilde{u}(t) = (u_0(t); \hat{u}_(t))^t = (u_0(t); u_1(t); \ldots; u_m(t))^t$$

satisfies equations, which are consequence of (1.40), (1.19), (1.31), (1.32), and also initial conditions:

$$\begin{aligned} &\frac{du_0}{dt} + A_0(u_0 + A_0^{-1/2}\sum_{k=1}^{m} Q_k^* u_k) = f(t), \qquad u_0(0) = u^0, \\ &\frac{du_k}{dt} + \gamma_k u_k - Q_k A_0^{1/2} u_0 = 0, \qquad u_k(0) = 0, \quad k = \overline{1,m}. \end{aligned} \tag{1.44}$$

Here each term is a continuous function of t with values in $\mathcal{H}$. Integrating (with respect to t from 0 to t) the last m equations, taking into account the initial condition, we get

$$u_k(t) = \int_0^t e^{-\gamma_k(t-s)} Q_k A_0^{1/2} u_0(s)\, ds, \qquad k = \overline{1,m}. \tag{1.45}$$

Substituting (1.45) in the first equation of (1.44), we obtain

$$\begin{gathered} \frac{du_0}{dt} + A_0(u_0 + \sum_{k=1}^{m} \int_0^t e^{-\gamma_k(t-s)} T_k u_0(s)\, ds) = f(t), \\ T_k := A_0^{-1/2} Q_k^* Q_k A_0^{1/2}, \qquad k = \overline{1,m}. \end{gathered} \tag{1.46}$$

Note that here, as stated above, the function

$$u_0(t) + \sum_{k=1}^{m} \int_0^t e^{-\gamma_k(t-s)} T_k u_0(s)\, ds =: v_0(t) \tag{1.47}$$

for all $t \in [0,T]$ belongs to $\mathcal{D}(A_0)$ $\;A_0 v_0(t) \in C[0,T;\mathcal{H}]$.

Note also that if $u_0 \in \mathcal{D}(A_0) = \mathcal{D}(A_k)$, then $A_0^{1/2} v_0 \in \mathcal{D}(A_0^{1/2}) = \mathcal{D}(A_k^{1/2})$, $k = \overline{1,m}$, and therefore by lemma 1.4

$$T_k u_0 = A_0^{-1/2} Q_k^* Q_k A_0^{1/2} u_0 = A_0^{-1/2} Q_k^+ A_k^{1/2} u_0 = A_0^{-1} A_k u_0 \in \mathcal{D}(A_0). \tag{1.48}$$

It follows from here that $T_k|\mathcal{D}(A_0)$ is a bounded operator acting in $\mathcal{D}(A_0)$.

Let us introduce in consideration the Hilbert space $\mathcal{H}_{A_0^2} := \mathcal{D}(A_0)$ with the norm equivalent to the norm of the graph: $\|u\|_{A_0^2} := \|A_0 u\|$, and consider the space $C[0,T;\mathcal{H}_{A_0^2}]$. Further, we consider relation (1.47) as an integral Volterra equation of the second order in the space $C[0,T;\mathcal{H}_{A_0^2}]$. Here $v_0(t) \in C[0,T;\mathcal{H}_{A_0^2}]$ is the given function, and the kernel (see (1.48))

$$T(t,s) := \sum_{k=1}^{m} e^{-\gamma_k(t-s)} T_k = \sum_{k=1}^{m} e^{-\gamma_k(t-s)} A_0^{-1} A_k \tag{1.49}$$

is continuous in t, s is an operator valued function acting in $\mathcal{H}_{A_0^2}$. By these properties, problem (1.47) has a unique solution $u_0(t) \in C[0,T;\mathcal{H}_{A_0^2}]$, and therefore each term in (1.47) is an element from $C[0,T;\mathcal{H}_{A_0^2}]$.

Hence the parentheses in equation (1.46) can be removed. Taking into account that operator A_0 boundedly acts from $C[0,T;\mathcal{H}_{A_0^2}]$ into $C[0,T;\mathcal{H}]$, we obtain equation (1.1) for the function $u_0(t) = u(t)$, where all the summands belong to $C[0,T;\mathcal{H}]$. ■

The following theorem is a corollary of theorem 1.6.

Theorem 1.7 *Suppose that in problem* (1.5) *the following conditions hold:*

$$\vec{u}^0(x) \in \mathcal{D}(A_0) = \vec{J}_0^2(\Omega), \quad \vec{f}(t,x) \in C^1[0,T;\vec{L}_2(\Omega)]. \tag{1.50}$$

Then this problem has a unique strong solution for $t \in [0,T]$ such that

$$\begin{gathered} \vec{u}(t,x) \in C[0,T;\vec{J}_0^2(\Omega)] \cap C^1[0,T;\vec{J}_0(\Omega)], \\ \nabla p(t,x) \in C[0,T;\vec{G}(\Omega)]. \end{gathered} \tag{1.51}$$

Proof. As stated in subsection 1.2, problem (1.5) is equivalent to the totality of problems (1.9)–(1.11). By (1.50), $P_0\vec{f}(t,x) \in C^1[0,T;\vec{J}_0(\Omega)]$, and $\vec{u}^0(x) \in \vec{J}_0^2(\Omega) = \mathcal{D}(A_0)$. Since (1.9)–(1.10) is a special case of problem (1.1) for $\mathcal{H} = \vec{J}_0(\Omega)$, $A_0 \mapsto \nu A_0$, $A_k \mapsto \nu\alpha_k A_0$ (A_0 in the right side is the Stokes operator), we get, by theorem 1.6, that problem (1.9)–(1.10) has a unique strong solution $\vec{u}(t,x) \in C[0,T;\vec{J}_0^2(\Omega)] \cap C^1[0,T;\vec{J}_0(\Omega)]$. In this case $\Delta\vec{u}(t,x) \in C[0,T;\vec{L}_2(\Omega)]$; then it follows from (1.11) that $\nabla p(t,x) \in C[0,T;\vec{G}(\Omega)]$. ■

Remark 1.8 In the formulation of theorem 1.6 the condition of continuous differentiability of the function $f(t)$ can be replaced by a different one, which ensures the strong solvability of problem (1.43). In particular it can be assumed that $f(t) \in C[0,T;\mathcal{H}]$, $f(t) \in \mathcal{D}(A_0)$, $A_0 f(t) \in C[0,T;\mathcal{H}]$ (see for instance [21, p. 166]).

Also, similar modifications can be done in the formulation of theorem 1.7.

2 Problem on normal oscillations. Indefinite approach

2.1 Passage to an equation with a bounded operator

By definition, normal oscillations are solutions of homogeneous problem (1.43) depending on t by the law

$$\tilde{u}(t) = \tilde{u} \exp(-\lambda t), \quad \tilde{u} \in \widetilde{\mathcal{H}}. \tag{2.52}$$

In order to determine amplitude elements $\tilde{u} \in \widetilde{\mathcal{H}}$ and eigenvalues $\lambda \in \mathbb{C}$, we substitute functions (2.52) into homogeneous equation (1.43) and get the spectral problem

$$\mathcal{A}\tilde{u} = \lambda \tilde{u}, \quad \tilde{u} \in \mathcal{D}(\mathcal{A}), \tag{2.53}$$

where $\mathcal{A}$ is an operator matrix defined by the formulas (1.37) – (1.40). Further, problem (2.53) is said to be a problem associated with evolutionary problem (1.1).

Let us formulate two evident properties of solutions of problem (2.53).

Property 1. The spectrum of problem (2.53) lies in the half-plane

$$\operatorname{Re} \lambda \geq c. \tag{2.54}$$

Property 2. The operator $\mathcal{A}$ has the bounded inverse operator $\mathcal{A}^{-1}$such that

$$\|\mathcal{A}^{-1}\| \leq c^{-1}. \tag{2.55}$$

Here $c > 0$ is the constant from condition (1.23) of uniform accretiveness of the operator $\mathcal{A}_0$. Note that after closing of $\mathcal{A}_0$ (lemma 1.3) inequality (1.23) holds also for the operator $\mathcal{A}$.

Using formulas (1.37) – (1.41), we pass from problem (2.53) to the equivalent problem

$$\mathcal{A}^{-1}\tilde{u} = \mu \tilde{u}, \quad \mu := \lambda^{-1}, \quad \tilde{u} = (u_0; \hat{u}_1)^t \in \widetilde{\mathcal{H}} = \mathcal{H}_0 \oplus \widehat{\mathcal{H}}_1, \tag{2.56}$$

with the bounded operator

$$\mathcal{A}^{-1} = \begin{pmatrix} I_0 & -A_0^{-1/2}Q_{10}^* \\ 0 & \hat{I}_1 \end{pmatrix} \begin{pmatrix} A_0^{-1} & 0 \\ 0 & (A_{11}+Q_{10}Q_{10}^*)^{-1} \end{pmatrix} \times$$

$$\times \begin{pmatrix} I_0 & 0 \\ Q_{10}A_0^{-1/2} & \hat{I}_1 \end{pmatrix} = \begin{pmatrix} A_0^{-1/2} & 0 \\ 0 & \hat{I}_1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} (I_0+Q_{10}^*A_{11}^{-1}Q_{10})^{-1} & -Q_{10}^*(A_{11}+Q_{10}Q_{10}^*)^{-1} \\ (A_{11}+Q_{10}Q_{10}^*)^{-1}Q_{10} & (A_{11}+Q_{10}Q_{10}^*)^{-1} \end{pmatrix} \begin{pmatrix} A_0^{-1/2} & 0 \\ 0 & \hat{I}_1 \end{pmatrix}. \tag{2.57}$$

2.2 General properties of solutions of the spectral problem

The further investigation of problems (2.53) and (2.56) conducted in this section is based on theory of linear operators that are selfadjoint in a Hilbert space with an indefinite metric. We recall here some concepts and facts of this theory (see [20]).

A Hilbert space $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ is called Krein space, or J-space if in this space along with scalar product $(\cdot,\cdot)$ there introduced an indefinite metric $[\cdot,\cdot]$ by the formula $[\cdot,\cdot] = (J\cdot,\cdot)$, where $J = P^+ - P^-$, $P^{\pm}$ are orthoprojectors onto $\mathcal{K}^{\pm}$, respectively, or, in other notation, $J = \mathrm{diag}\,(I^+; -I^-)$.

With respect to the indefinite metric every vector x of the space assume the definite sign: it is positive ($[x,x] > 0$), negative ($[x,x] < 0$) or neutral ($[x,x] = 0$). In a natural way, signs of the subspaces and concept of order are introduced. Thus, for example, a subspace $\mathcal{L}_+$ is called nonnegative if $[x,x] \geq 0$, $\forall x \in \mathcal{L}_+$, and maximal nonnegative if it is not a principal part of none other nonnegative subspace.

In the sequel, we often will use the following fact: a subspace $\mathcal{L}_+$ is maximal nonnegative if there exists a restriction $K_+ : \mathcal{K}^+ \to \mathcal{K}^-$ such that $\mathcal{L}_+ = \{x = x_+ + K_+x_+ \mid x_+ \in \mathcal{K}^+\}$. This restriction K_+ is called an angular operator of the subspace $\mathcal{L}_+$.

A positive subspace $\mathcal{L}_+$ is said to be uniformly positive if it is a Hilbert space with respect to the scalar product generated by the indefinite metric.

We shall say that the subspace $\mathcal{L}_+$ belongs to the class $\mathbf{h}^+$ $\mathcal{L}_+ \in \mathbf{h}^+$ if it can be represented as the sum of a uniformly positive subspace and finite-dimensional neutral one. In particular $\mathcal{L}_+ \in \mathbf{h}^+$ if its angular operator is compact.

In the same way, nonpositive and maximal nonpositive subspaces, their angular operators, uniformly negative subspaces and subspaces of the class $\mathbf{h}^-$ are introduced.

If the maximal nonnegative subspace $\mathcal{L}_+$ and the maximal nonpositive one $\mathcal{L}_-$ are J-orthogonal, i. e; $[x_+, x_-] = 0$, $x_\pm \in \mathcal{L}_\pm$, then we say that $\mathcal{L}_+$ and $\mathcal{L}_-$ form a dual pair $\{\mathcal{L}_+, \mathcal{L}_-\}$. We shall write $\{\mathcal{L}_+, \mathcal{L}_-\} \in \mathbf{h}$ if $\mathcal{L}_\pm \in \mathbf{h}^\pm$.

Let T be a linear densely defined operator in $\mathcal{K}$. In a usual way, definitions of J-conjugate operator T^c, J-symmetric operator ($T \subset T^c$) and J-selfadjoint operator ($T = T^c$) are introduced. It follows from the connection $T^c = JT^*J$ of the operators T^c and T^* that in particular the spectrum of J-selfadjont operator is symmetric.

We say that a continuous J-selfadjoint operator T belongs to the class $\mathbf{H}$ if there exists at least one dual pair $\{\mathcal{L}_+, \mathcal{L}_-\} \in \mathbf{h}$ of invariant with respect to T subspaces and every T-invariant dual pair belongs to the class $\mathbf{h}$.

In the sequel, the space $\widetilde{\mathcal{H}}$ will be taken for $\mathcal{K}$.

Lemma 2.1 *The operator matrices $\mathcal{A}$ (1.37) – (1.40) and $\mathcal{A}^{-1}$ from (2.57) are J-selfadjoint with*

$$J := \operatorname{diag}(I_0; -\hat{I}_1). \tag{2.58}$$

The spectrum of the operator $\mathcal{A}$ is symmetric with respect to the real axis and lies in the half-plane $\operatorname{Re}\lambda \geq c > 0$, where c is the constant from inequality (1.23).

Proof. The property of J-symmetry of the operators $\mathcal{A}$ and $\mathcal{A}^{-1}$ can be checked directly, using representations (1.37), (1.38), (1.41), (2.57). Since $\mathcal{A}^{-1}$ is a bounded operator, it is J-selfadjoint. Therefore its inverse $\mathcal{A}$ is also J-selfadjoint. Hence, as was noted above, the spectra of the operators $\mathcal{A}$ and $\mathcal{A}^{-1}$ is symmetric with respect to the real axis. Finally, the property $\operatorname{Re}\lambda \geq c > 0$, $\lambda \in \sigma(\mathcal{A})$ is stated above. ■

Lemma 2.2 *The nonreal spectrum of the operator $\mathcal{A}$ consists of at most finite number of eigenvalues with regard for their algebraic multiplicity. In particular the spectrum of problem (2.5), and hence problem (2.2), is positive except for at most finite number (with regard for algebraic multiplicity) of nonreal eigenvalues.*

Proof. According to (2.57), the operator matrix

$$\mathcal{A}^{-1} := \begin{pmatrix} \mathcal{A}_{00}^{(-1)} & \mathcal{A}_{01}^{(-1)} \\ \mathcal{A}_{10}^{(-1)} & \mathcal{A}_{11}^{(-1)} \end{pmatrix} =$$

$$= \begin{pmatrix} A_0^{-1/2}(I_0 + Q_{10}^* A_{11}^{-1} Q_{10})^{-1} A_0^{-1/2} & -A_0^{-1/2} Q_{10}^* (A_{11} + Q_{10} Q_{10}^*)^{-1} \\ (A_{11} + Q_{10} Q_{10}^*)^{-1} Q_{10} A_0^{-1/2} & (A_{11} + Q_{10} Q_{10}^*)^{-1} \end{pmatrix} \tag{2.59}$$

has as its entries compact operators $\mathcal{A}_{00}^{(-1)}$, $\mathcal{A}_{01}^{(-1)}$ and $\mathcal{A}_{10}^{(-1)} = -\big(\mathcal{A}_{01}^{(-1)}\big)^*$, and also a bounded positive definite operator $\mathcal{A}_{11}^{(-1)}$. In particular, by virtue of the property $\mathcal{A}_{01}^{(-1)} \in \mathfrak{S}_\infty$ by H. Langer's theorem (see [20], p. 217), J-selfadjoint operator $\mathcal{A}^{-1}$ has the invariant dual pair $\{\mathcal{L}_+, \mathcal{L}_-\}$. Let $K = K_+ : \mathcal{H}_0 \to \widehat{\mathcal{H}}_1$ be an angular operator of subspace $\mathcal{L}_+$, i. e; $\|K\| \le 1$ and

$$\mathcal{L}_+ = \{\tilde{u} := (u_0; \hat{u}_1)^t \in \widetilde{\mathcal{H}} = \mathcal{H}_0 \oplus \widehat{\mathcal{H}}_1 \colon\ \hat{u}_1 = K u_0,\ \forall u_0 \in \mathcal{H}_0\}. \tag{2.60}$$

For any $\tilde{u} = (u_0; K u_0)^t \in \mathcal{L}_+$ we have $\mathcal{A}^{-1}\tilde{u} \in \mathcal{L}_+$. From here and from (2.59) an equation for K can be deduced in a standard way:

$$\begin{gathered} (A_{11} + Q_{10}Q_{10}^*)K = K A_0^{-1/2}(I_0 + Q_{10}^* A_{11}^{-1} Q_{10})^{-1} A_0^{-1/2} - \\ -(A_{11} + Q_{10}Q_{10}^*)^{-1} Q_{10} A_0^{-1/2} - K A_0^{-1/2} Q_{10}^* (A_{11} + Q_{10}Q_{10}^*)^{-1} K. \end{gathered} \tag{2.61}$$

Since here $A_0^{-1/2} \in \mathfrak{S}_\infty$ and the operator $A_{11} + Q_{10}Q_{10}^*$ has the bounded inverse,

$$K \in \mathfrak{S}_\infty. \tag{2.62}$$

Therefore $\{\mathcal{L}_+, \mathcal{L}_-\} \in \mathbf{h}$ that implies the inclusion $\mathcal{A} \in \mathbf{H}$. Now conclusions of the lemma follow directly from [20, p. 245]. ■

Note that for $\|K\| < 1$ the operator $\mathcal{A}$ is similar to a selfadjoint one and hence in this case its spectrum and the spectrum of problems (2.5) and (2.2) are positive.

Theorem 2.3 *Let $B := A_{11} + Q_{10}Q_{10}^*$. Then essential (limit) spectrum $\sigma_{\mathrm{ess}}(\mathcal{A})$ of the operator $\mathcal{A}$ coincides with the set $\{\infty\} \cup \sigma_{\mathrm{ess}}(B)$.*

Proof. By proved above positive definiteness and continuity of the operator B it is sufficient to state that essential spectrum of the operator $\mathcal{A}^{-1}$ coincides with the set $\{0\} \cup \sigma_{\mathrm{ess}}(B^{-1})$. The last conclusion follows directly from the known Weyl's theorem as (see (2.59)) $\mathcal{A}^{-1}$ is a compact perturbation of the operator $\mathcal{F} := \mathrm{diag}\,(0; B^{-1})$, and (see lemma 2.1) $\rho(\mathcal{F}) \cap \rho(\mathcal{A}^{-1}) \neq \emptyset$. ■

It will be clear from the sequent that the spectrum of the operator $\mathcal{A}$, and hence of problem (2.53), can be both discrete with a finite number of limit points on $\mathbb{R}_+$ and containing as limit spectrum some segments. It turns out that structure of the spectrum of problem (2.53) depends essentially on to what extent the operators A_k of problem (1.1) are close by their properties to each other.

Theorem 2.4 *In problem* (2.53) *for* $m = 1$ *we can select commuting operators* A_0 *and* A_1 *so that the spectrum* $\sigma(B)$ *of the operator* B *is its essential spectrum* ($\sigma(B) = \sigma_{\rm ess}(B)$) *and coincides with the segment* $[\gamma_1 + 1; \gamma_1 + 2]$. *For* $m = 2$, $\gamma_1 = \gamma_2$ *it is possible to select commuting operators* A_0, A_1 *and* A_2 *such that* $\sigma(B) = \sigma_{\rm ess}(B) = \{\gamma_1\} \cup [\gamma_1 + 2; \gamma_1 + 4]$.

Proof. First consider the case $m = 1$. Then (see (1.31) – (1.33), (1.19))

$$B = A_{11} + Q_{10}Q_{10}^* = \gamma_1 I_1 + (A_1^{1/2}A_0^{-1/2})(A_1^{1/2}A_0^{-1/2})^*. \tag{2.63}$$

Let us select operators A_0 and A_1 in order to they commute and the set of accumulation points of the spectrum of the operator $(A_1^{1/2}A_0^{-1/2})(A_1^{1/2}A_0^{-1/2})^*$ coincides with the segment $[1; 2]$. Let $\lambda_1 \geq \lambda_2 \geq \ldots > 0$ is the ordered set of eigenvalues of the operator A_1^{-1} and every eigenvalue is prime. Since the operator A_1^{-1} and operator A_0^{-1} must be commuting, eigenvalues of A_1^{-1} must be eigenvalues for A_0^{-1} and also for $(A_1^{1/2}A_0^{-1/2})(A_1^{1/2}A_0^{-1/2})^*$.

Choose desired countable set of eigenvalues of the operator $(A_1^{1/2}A_0^{-1/2})(A_1^{1/2}A_0^{-1/2})^*$ by the following rule: $\alpha_{01} = 1$; $\alpha_{11} = 1$, $\alpha_{12} = 2$; $\alpha_{21} = 1$, $\alpha_{22} = 3/2$, $\alpha_{23} = 2$; .... Here in the every next group we add one element that are half-sum of the neighboring elements; for instance $\alpha_{31} = 1$, $\alpha_{32} = 5/4$, $\alpha_{33} = 3/2$, $\alpha_{34} = 2$; $\alpha_{41} = 1$, $\alpha_{42} = 5/4$, $\alpha_{43} = 3/2$, $\alpha_{44} = 7/4$, $\alpha_{45} = 2$, etc.

Enumerate these numbers; we get a sequence $\{\beta_n\}_{n=1}^{\infty}$. We suppose $\mu_n := \beta_n\lambda_n$ and will assume that μ_n are eigenvalues of the operator A_0^{-1}, which correspond to the same eigen elements that eigenvalues λ_n for the operator A_1^{-1}. Then the operator $(A_1^{1/2}A_0^{-1/2})(A_1^{1/2}A_0^{-1/2})^*$ is diagonal in the basis from mentioned eigen elements and its eigenvalues $\{\beta_n\}$ form a dense set on the segment $[1; 2]$, and each eigenvalue β_n is infinite-to-one by construction, i. e; it belongs to the set $\sigma_{\rm ess}\big((A_1^{1/2}A_0^{-1/2})(A_1^{1/2}A_0^{-1/2})^*\big)$. By representation (2.63) of the operator B we get that $\sigma(B) = \sigma_{\rm ess}(B) = [\gamma_1 + 1; \gamma_1 + 2]$.

Similar construction can be done for $m = 2$, $\gamma_1 = \gamma_2$, $A_2 = A_1$. Introducing again numbers λ_n, β_n μ_n, and also the operator $(A_1^{1/2}A_0^{-1/2})(A_1^{1/2}A_0^{-1/2})^* = (A_2^{1/2}A_0^{-1/2})(A_2^{1/2}A_0^{-1/2})^*$, we get that its eigenvalues β_n form a dense set on the segment $[1; 2]$. Since in this case

$$B = \operatorname{diag}(\gamma_1 I; \gamma_1 I) + (\gamma_{ik})_{i,k=1}^2, \quad \gamma_{ik} := (A_i^{1/2}A_0^{-1/2})(A_k^{1/2}A_0^{-1/2})^*, \tag{2.64}$$

the operator B can be represented in the form

$$B = \overset{\infty}{\underset{n=1}{\oplus}} B_n, \quad B_n = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_1 \end{pmatrix} + \begin{pmatrix} \beta_n & \beta_n \\ \beta_n & \beta_n \end{pmatrix}.$$

Therefore eigenvalues of the operator B_n for any n equal γ_1 and $\gamma_1 + 2\beta_n$ respectively. Hence all the spectrum $\sigma(B)$ of the operator $B = \overset{\infty}{\underset{n=1}{\oplus}} B_n$ is an essential spectrum and coincides with $\{\gamma_1\} \cup [\gamma_1 + 2; \gamma_1 + 4]$. ■

Conducted in the proof of theorem 2.4 reasoning show how to construct examples of problems of the form (2.53) with sufficiently complicated structure of the essential spectrum in the case of commuting operators A_k. On the other hand, in the next section we shall mention (see theorem 3.2) sufficient conditions for the spectrum of problem (2.53) being discrete with a finite number of limit points on $\mathbb{R}_+$. The existence of a branch of he discrete spectrum with the limit point $+\infty$ can be also stated in the general case.

Definition 2.5 *We say that an operator $A \in \mathfrak{S}_\infty$ belongs to a class $\mathfrak{S}_p$ if its s-numbers, i. e; eigenvalues of the operator $(A^*A)^{1/2}$, are summable with the p-th power:*

$$\sum_{k=1}^{\infty} s_k^p(A) = \sum_{k=1}^{\infty} [\lambda_k((A^*A)^{1/2})]^p < \infty. \tag{2.65}$$

Definition 2.6 *A basis $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is called a Riesz basis if $\psi_n = T\varphi_n$, where $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}$, and T, $T^{-1} \in \mathcal{L}(\mathcal{H})$. A Riesz basis is called p-basis* (see [26]) *if $T = I + T_1$, $T_1 \in \mathfrak{S}_p$.*

Theorem 2.7 *Problem* (2.53) *has a countable set of positive eigenvalues $\{\lambda_n^{(\infty)}\}_{n=1}^{\infty}$ with a unique limit point $\lambda = +\infty$ and eigen elements $\{\tilde{u}_n^{(\infty)}\}_{n=1}^{\infty}$, $\tilde{u}_n^{(\infty)} = (u_{0n}^{(\infty)}; \hat{u}_{1n}^{(\infty)})^t \in \mathcal{H}_0 \oplus \widehat{\mathcal{H}}_1 = \widetilde{\mathcal{H}}$ such that their projections $\{u_{0n}^{(\infty)}\}_{n=1}^{\infty}$ onto $\mathcal{H}_0 = \mathcal{H}$ form a Riesz basis with a finite defect in the space $\mathcal{H}$. If the following condition holds*

$$A_0^{-1} \in \mathfrak{S}_{p_0}, \tag{2.66}$$

then mentioned Riesz basis is p_0-basis (with a finite defect) in $\mathcal{H}$.

Proof. Since eigenvalues of an operator and its inverse are mutually inverse, and they have the same eigen elements, it is sufficient to check whether the operator $\mathcal{A}^{-1}$ has a countable set of eigenvalues with a unique limit point $\mu = 0$ and with corresponding properties of eigen elements. In this connection we consider the introduced in the proof of lemma 2.2 a maximal nonnegative subspace $\mathcal{L}_+ \subset \widetilde{\mathcal{H}}$, which is invariant under $\mathcal{A}^{-1}$, and also corresponding to this subspace an angular operator $K = K_+$ (see (2.60)).

We proceed to show that the restriction $\mathcal{A}^{-1} \mid \mathcal{L}_+$ of the operator $\mathcal{A}^{-1}$ to $\mathcal{L}_+$ is compact operator acting in $\mathcal{L}_+$. Indeed, by (2.59), for any element $\tilde{u} = (u_0; K_+ u_0)^t \in \mathcal{L}_+$ the operator $\mathcal{A}^{-1} \mid \mathcal{L}_+$ acts by the law

$$(\mathcal{A}^{-1} \mid \mathcal{L}_+)\tilde{u} = \big((\mathcal{A}_{00}^{(-1)} + \mathcal{A}_{01}^{(-1)} K)u_0; (\mathcal{A}_{10}^{(-1)} + \mathcal{A}_{11}^{(-1)} K)u_0\big)^t. \tag{2.67}$$

Let $(P_+ \mid \mathcal{L}_+)$ is the restriction to $\mathcal{L}_+$ of the orthoprojector P_+ acting by the law $P_+\tilde{u} := (u_0; 0)^t$, $\forall \tilde{u} \in \widetilde{\mathcal{H}}$. Then taking into account the equality $u_0 = (P_+ \mid \mathcal{L}_+)\tilde{u}$, from (2.67) we get

$$(P_+ \mid \mathcal{L}_+)(\mathcal{A}^{-1} \mid \mathcal{L}_+) = (\mathcal{A}_{00}^{(-1)} + \mathcal{A}_{01}^{(-1)} K)(P_+ \mid \mathcal{L}_+), \tag{2.68}$$

by the arbitrariness of $\tilde{u} \in \widetilde{\mathcal{H}}$.

Since (see for instance [20, . 37]), the operator $(P_+ \mid \mathcal{L}_+)$ maps homeomorphically $\mathcal{L}_+$ onto $\widetilde{\mathcal{H}}_+ = \mathcal{H}_0 = \mathcal{H}$, it follows from (2.68) that the operator $(\mathcal{A}^{-1} \mid \mathcal{L}_+)$ is similar to the operator $\mathcal{A}_{00}^{(-1)} + \mathcal{A}_{01}^{(-1)} K$:

$$(\mathcal{A}^{-1} \mid \mathcal{L}_+) = (P_+ \mid \mathcal{L}_+)^{-1}(\mathcal{A}_{00}^{(-1)} + \mathcal{A}_{01}^{(-1)} K)(P_+ \mid \mathcal{L}_+). \tag{2.69}$$

As the angular operator K is bounded and the operators $\mathcal{A}_{00}^{(-1)}$, $\mathcal{A}_{01}^{(-1)}$ are compact the operator $\mathcal{A}_{00}^{(-1)} + \mathcal{A}_{01}^{(-1)} K$, and hence the operator $(\mathcal{A}^{-1} \mid \mathcal{L}_+)$, is compact. Since $\mathcal{L}_+$ is nonnegative subspace of the class $\mathbf{h}^+$, its neutral vectors form finite-dimensional linear subspace $\mathcal{L}_+^\circ$ being invariant by $\mathcal{A}^{-1}$. Suppose $\{\nu_k\}_{k=1}^n = \sigma(\mathcal{A}^{-1} \mid \mathcal{L}_+^\circ)$. Then $\mathcal{L}_+$ can be represented as direct sum of two invariant by $\mathcal{A}^{-1}$ subspaces:

$$\mathcal{L}_+ = \mathrm{l.h.}\,\{\mathcal{L}_\nu(A) \mid \nu \in \sigma(\mathcal{A}^{-1} \mid \mathcal{L}_+^\circ)\} \dotplus \mathcal{L},$$

and $\sigma(\mathcal{A}^{-1} \mid \mathcal{L}) = \sigma(\mathcal{A}^{-1} \mid \mathcal{L}_+) \setminus \sigma(\mathcal{A}^{-1} \mid \mathcal{L}_+^\circ)$; here $\mathcal{L}_\nu(A)$ is a root subspace of the operator $(\mathcal{A}^{-1} \mid \mathcal{L}_+)$ corresponding to its eigenvalue ν, and by l.h. we denote the linear hull of the corresponding elements. Since $\mathcal{L}_+^\circ \subset \mathrm{l.h.}\,\{\mathcal{L}_\nu(A) \mid \nu \in \sigma(\mathcal{A}^{-1} \mid \mathcal{L}_+^\circ)\}$, $\mathcal{L}$ is a Hilbert space with respect to a scalar product inducing by the indefinite metric, and it has a finite codimension $\dim \mathrm{l.h.}\,\{\mathcal{L}_\nu(A) \mid \nu \in \sigma(\mathcal{A}^{-1} \mid \mathcal{L}_+^\circ)\}$ in $\mathcal{L}_+$. Therefore in $\mathcal{L}$ there exists unconditional basis $\{\tilde{u}_n^{(\infty)}\}_{n=1}^\infty$, $\tilde{u}_n^{(\infty)} = (u_{0n}^{(\infty)}; \hat{u}_{1n}^{(\infty)})^t \in \mathcal{H}_0 \oplus \widehat{\mathcal{H}}_1 = \widetilde{\mathcal{H}}$ which consists of eigenvalues of the operator $\mathcal{L}^{-1}$ and it has the same finite codimension in $\mathcal{L}_+$. Its projection $\{u_{0n}^{(\infty)}\}_{n=1}^\infty$ $\mathcal{H}_0 = \mathcal{H}$ forms the Riesz basis with the same finite defect in the space $\mathcal{H}$, because, by above, the operator $(P_+ \mid \mathcal{L}_+)$ maps homeomorphically $\mathcal{L}_+$ onto $\mathcal{H}$.

Suppose now that condition (2.66) also holds. Then it follows from (2.61) that the angular operator K of the subspace $\mathcal{L}_+$ belongs to the class $\mathfrak{S}_{2p_0}$ and hence (cf. [20, p. 271]) $\{\tilde{u}_n^{(\infty)}\}_{n=1}^\infty$ is $2p_0$-basis in $\mathcal{L}$. Therefore [27, p. 55], $\{u_{0n}^{(\infty)}\}$ is p_0-basis in its closed linear hull. ■

2.3 An important special case

Consider the following situation. Suppose that in problem (2.53) operators A_k ($k = \overline{1,m}$) forming the matrix operator $\mathcal{A}$ by formulas (1.19), (1.31) – (1.33), (1.37) – (1.40) satisfy the conditions

$$A_k = \alpha_k A_0, \quad \alpha_k > 0 \quad (k = \overline{1,m}). \tag{2.70}$$

This simple, but important special case allows to investigate spectral problem (2.53) completely enough.

Theorem 2.8 *If conditions* (2.70) *hold, then spectrum of the operator* $\mathcal{A}$ *is discrete with the exception of* ∞ *and* m *finite points of accumulation* $\{\beta_k\}_{k=1}^m$*, which are zeroes of the function*

$$e_0(\lambda) := 1 + \sum_{k=1}^{m} \alpha_k/(\gamma_k - \lambda) \tag{2.71}$$

and satisfy the condition

$$0 < \gamma_1 < \beta_1 < \ldots < \gamma_m < \beta_m < \infty. \tag{2.72}$$

In this case $\sigma(\mathcal{A}) \setminus (\{\infty\} \cup \{\beta_k\}_{k=1}^m)$ *consists of at most finite number of nonreal eigenvalues and* $m+1$ *branches of positive eigenvalues with one of branches tending to* $+\infty$*, and others* m *to* $\{\beta_k\}_{k=1}^m = \sigma_{\text{ess}}(\mathcal{A}) \setminus \{+\infty\}$*.*

Moreover, if $\{\tilde{u}_n\}_{n=1}^\infty$*,* $\tilde{u}_n = (u_{0n}; \hat{u}_{1n})^t \in \mathcal{H}_0 \oplus \widehat{\mathcal{H}}_1 = \widetilde{\mathcal{H}}$*, are eigen elements of the operator* $\mathcal{A}$ *corresponding to the branch of eigenvalues tending to* $+\infty$ *(to the branches tending to* $\{\beta_k\}_{k=1}^m$*,respectively), then their projections* $\{u_{0n}\}_{n=1}^\infty$ *onto* $\mathcal{H}_0 = \mathcal{H}$ *form a Riesz basis with a finite defect in the space* $\mathcal{H}$ *(projections* $\{\hat{u}_{1n}\}_{n=1}^\infty$ *onto* $\widehat{\mathcal{H}}_1$ *form a Riesz basis with a finite defect in the space* $\widehat{\mathcal{H}}_1$*, respectively). If the condition* $A_0^{-1} \in \mathfrak{S}_{p_0}$ *holds, then the mentioned Riesz bases are* p_0*-bases (with a finite defect) in* $\mathcal{H}$ *and* $\widehat{\mathcal{H}}_1$*, respectively.*

Proof. First note that the conclusions on nonreal spectrum and on the branch tending to ∞ are proved, for the general case, in lemma 2.2 and theorem 2.7, respectively.

Under assumptions (2.70), by formulas (1.31) – (1.33), we have

$$\begin{gathered} Q_k = \overline{Q_k^+} = \alpha_k^{1/2} I \quad (k = \overline{1,m}), \quad Q_{10} = (\alpha_1^{1/2} I; \ldots; \alpha_m^{1/2} I)^t, \\ Q_{10}^* = (\alpha_1^{1/2} I; \ldots; \alpha_m^{1/2} I). \end{gathered} \tag{2.73}$$

On the one hand, theorem 2.3 asserts that $\sigma_{\text{ess}}(\mathcal{A}) = \{\infty\} \cup \sigma_{\text{ess}}(B)$, where

$$B = A_{11} + Q_{10} Q_{10}^* = (\gamma_k I)_{k=1}^m + \|\alpha_i^{1/2} \alpha_j^{1/2} I\|_{i,j=1}^m. \tag{2.74}$$

Therefore $\sigma_{\text{ess}}(B) = \sigma(B)$ consists of at most m different eigenvalues of infinite multiplicity.

On the other hand, by (1.38) – (1.40), problem (2.53) takes the form

$$A_0(u_0 + \sum_{k=1}^{m} \alpha_k^{1/2} A_0^{-1/2} u_k) = \lambda u_0, \quad u_0 + \sum_{k=1}^{m} \alpha_k^{1/2} A_0^{-1/2} u_k \in \mathcal{D}(A_0), \quad (2.75)$$

$$-\alpha_k^{1/2} A_0^{1/2} u_0 + \gamma_k u_k = \lambda u_k \quad (k = \overline{1,m}). \quad (2.76)$$

Let us check whether the numbers $\lambda = \gamma_j$ $(j = \overline{1,m})$ are regular for the operator $\mathcal{A}$. Since the spectrum of this operator is real, with the exception of perhaps finite number of eigenvalues, it is sufficient to check that γ_j $(j = 1,m)$ are not boundary points of the spectrum of $\mathcal{A}$. Indeed, let $\widetilde{u}^{(n)} = (u_0^{(n)}; u_1^{(n)}, \ldots, u_m^{(n)})^t \in \mathcal{D}(\mathcal{A})$ and $(\mathcal{A} - \gamma_j \mathcal{I})\widetilde{u}^{(n)} \to 0$, $n \to \infty$. This relation can be write as the system:

$$A_0(u_0^{(n)} + \sum_{k=1}^{m} \alpha_k^{1/2} A_0^{-1/2} u_k^{(n)}) - \gamma_j u_0^{(n)} \to 0, \quad (2.77)$$

$$-\alpha_k^{1/2} A_0^{1/2} u_0^{(n)} + \gamma_k u_k^{(n)} - \gamma_j u_k^{(n)} \to 0 \quad (k = \overline{1,m}). \quad (2.78)$$

Then from(2.78) for $k = j$ we get $A_0^{1/2} u_0^{(n)} \to 0$ and hence $u_0^{(n)} \to 0$. Therefore for $k \neq j$ from (2.78) we have $u_k^{(n)} \to 0$. Applying from the left to (2.77) the operator $A_0^{-1/2}$, we obtain $u_j^{(n)} \to 0$. Consequently, $\widetilde{u}^{(n)} \to 0$. Thus $\lambda = \gamma_j$ $(j = \overline{1,m})$ are not boundary points of the spectrum, i. e; they are regular.

Excluding in (2.75), (2.76) the elements u_k (for $\lambda \neq \gamma_k$), we get the problem

$$e_0(\lambda) u_0 = \lambda A_0^{-1} u_0, \quad u_0 \in \mathcal{H}, \quad (2.79)$$

where the function $e_0(\lambda)$ is defined by formula (2.71). It follows from this that eigen elements of problem (2.79) coincide with eigen elements $\{u_n(A_0)\}_{n=1}^{\infty}$ of the operator A_0, and eigenvalues λ are solutions of series of characteristic equations

$$e_0(\lambda) = \lambda / \lambda_n(A_0), \quad n \in \mathbb{N}, \quad (2.80)$$

where $\{\lambda_n(A_0)\}_{n=1}^{\infty}$ are eigenvalues (of finite multiplicity) of A_0, $0 < \lambda_1(A_0) \leq \ldots \leq \lambda_n(A_0) \leq \ldots$, $\lambda_n(A_0) \to +\infty$ $(n \to \infty)$.

Consider equation (2.80). The function $e_0(\lambda)$ in the left-hand side increases on its domain of definition, with values from 1 to $+\infty$ on $(-\infty, \gamma_1)$, from $-\infty$ to $+\infty$ on each of intervals (γ_j, γ_{j+1}), $j = \overline{1, m-1}$ and from $-\infty$ to 1 on $(\gamma_m, +\infty)$; the function in the right-hand side is linear with the coefficient $1/\lambda_n(A_0) \to 0$. Hence equation (2.80) for any n has $m - 1$ real roots, and beginning with a certain $n = n_0$ – $(m + 1)$ real roots. For $n \geq n_0$ all the roots of problem (2.80) fall into m series with the limit points $\{\beta_k\}_{k=1}^{m}$, $e_0(\beta_k) = 0$, $(k = \overline{1,m})$, and also series of eigenvalues with the limit point $+\infty$. Zeroes β_k of the function $e_0(\lambda)$

satisfy condition (2.72), and corresponding series tend monotonnically to these zeroes from the right.

Therefore $\{\beta_k\}_{k=1}^m$ is a part of the spectrum of B, and hence

$$\sigma(B) = \{\beta_k\}_{k=1}^m. \tag{2.81}$$

It remains to verify the assertions on basicity of projections of the eigen vectors. It can be done by the reasoning similar to that in the proof of theorem 2.7. In this case we consider the maximal nonpositive invariant subspace instead of the maximal nonnegative invariant one, and take into account that $\beta_k,\ k=\overline{1,m}$ are not eigenvalues of $\mathcal{A}$ (see (2.80)). ■

Theorems 2.4 and 2.8 show that properties of the spectrum of problem (2.53) depend essentially on how operator coefficients A_k ($k=\overline{1,m}$) are close or far from each other by their properties. Additional properties of solutions of problem (2.53) can be established by means of methods of theory of operator-valued functions depending on spectral parameter (see for instance [28]).

Remark 2.9 The dimension of the space $\mathcal{H}$ has an influence on multiplicity, but not on eigenvalues of B playing an important role in the proof of theorem 2.8. Therefore for $\dim\mathcal{H}=1$ we have $\det(B-\lambda I)=\prod_{k=1}^m(\gamma_k-\lambda)\,e_0(\lambda)$. Indeed, from the left and from the right there are polynomials with the same roots and the same leading coefficient.

3 Problem on normal oscillations. Applying of the spectral theory of operator pencils

3.1 On existence of branches of eigenvalues

Consider a situation, which is slightly more difficult than that in subsection 2.3, and corresponds to its weak perturbation.

As a preliminary we give the following

Definition 3.1 *We shall say that operator-valued function $L(\lambda)$, $\lambda\in\mathbb{C}$, with values in $\mathcal{L}(\mathcal{H})$ is selfadjoint if $(L(\overline{\lambda}))^*=L(\lambda)$.*

Suppose that in (2.53) for operators A_k

$$Q_k^*Q_k=\alpha_k I+F_k, \quad F_k=F_k^*\in\mathfrak{S}_\infty. \tag{3.82}$$

For definiteness, from now on we shall denote the corresponding operator $\mathcal{A}$ (see (1.37)–(1.40)) and its derivatives by $(\cdot)(F)$, for example, $\mathcal{A}(F)$, $A_{11}(F)$ etc.

Points β_j ($j=\overline{1,m}$), as above, are zeroes of the function $e_0(\lambda)$ (2.71).

Suppose

$$A(\lambda) := e_0(\lambda)I + \sum_{k=1}^{m}(\gamma_k - \lambda)^{-1}F_k - \lambda A_0^{-1}. \tag{3.83}$$

Then $A(\lambda)$ is a holomorphic in neighborhoods of points the β_j $(j = \overline{1,m})$ operator-valued function and

$$A'(\lambda) = e_0'(\lambda)I + \sum_{k=1}^{m}(\gamma_k - \lambda)^{-2}F_k - A_0^{-1} \tag{3.84}$$

is its derivative.

Theorem 3.2 *Suppose condition* (3.82) *holds. Then*

$$\sigma(\mathcal{A}(F)) \setminus (\{+\infty\} \cup \{\beta_j\}_{j=1}^{m})$$

is a discrete set forming $m+1$ branches

$$\{\{\lambda_k^{(j)}\}_{k=1}^{\infty}\}_{j=1}^{m} \cup \{\lambda_k^{(\infty)}\}_{k=1}^{\infty}$$

of positive eigenvalues with the limit points $\{\beta_k\}_{k=1}^{m}$ and $+\infty$ respectively. It also includes a finite number of nonreal eigenvalues.

In this case if $\{\tilde{u}_n^{(\infty)}\}_{n=1}^{\infty}$, $\tilde{u}_n^{(\infty)} = (u_{0n}^{(\infty)}; \hat{u}_{1n}^{(\infty)})^t \in \mathcal{H}_0 \oplus \widehat{\mathcal{H}}_1 = \widetilde{\mathcal{H}}$ ($\{\tilde{u}_n^{(j)}\}_{n=1}^{\infty}$, $\tilde{u}_n^{(j)} = (u_{0n}^{(j)}; \hat{u}_{1n}^{(j)})^t \in \mathcal{H}_0 \oplus \widehat{\mathcal{H}}_1 = \widetilde{\mathcal{H}}$ respectively) are eigenvalues of $\mathcal{A}$ corresponding to the branch $\{\lambda_k^{(\infty)}\}_{k=1}^{\infty}$ of eigenvalues tending to $+\infty$ (to the branches $\{\lambda_k^{(j)}\}_{k=1}^{\infty}$ tending to $\{\beta_j\}$, $j = \overline{1,m}$ respectively), then their projections $\{u_{0n}^{(\infty)}\}_{n=1}^{\infty}$ onto $\mathcal{H}_0 = \mathcal{H}$ form a Riesz basis with a finite defect in the space $\mathcal{H}$ (the closed linear hull of projections $\{\hat{u}_{1n}^{(j)}\}_{n=1}^{\infty}$, $j = \overline{1,m}$, onto $\widehat{\mathcal{H}}_1$ – subspace of finite codimension in $\widehat{\mathcal{H}}_1$ respectively).

Under the additional condition

$$\ker(\mathcal{A}(F) - \beta_j\mathcal{I}) = \{0\}, \quad (j = \overline{1,m}) \tag{3.85}$$

the system $\{\hat{u}_{1n}^{(j)}\}_{n=1}^{\infty}$, $j = \overline{1,m}$, forms a Riesz basis in its closed linear hull.

If

$$A'(\beta_j) \gg 0, \quad (j = \overline{1,m}), \tag{3.86}$$

then (without assumption (3.85)*) projections of $\{\hat{u}_{1n}^{(j)}\}_{n=1}^{\infty}$ onto $\mathcal{H}_j$, $j = \overline{1,m}$, form a Riesz basis with a finite defect in $\mathcal{H}_j = \mathcal{H}$, $j = \overline{1,m}$.*

Proof. As in the proof of theorem 2.8, we note that conclusions on the nonreal spectrum and on the branch tending to $+\infty$ are proved for the general case in lemma 2.2 and theorem 2.7 respectively.

It follows from (3.82) that there exists a unitary operator U_j such that $Q_j(F) = U_j(\alpha_j I + F_j)^{1/2}$, $j = \overline{1,m}$. Therefore,

$$Q_j(F) = \alpha_j^{1/2} U_j + D_j, \qquad D_j \in \mathfrak{S}_\infty, \quad j = \overline{1,m}. \tag{3.87}$$

Consider the operator $B(F) := A_{11}(F)+Q_{10}(F)Q_{10}(F)^*$. It follows from (1.19), (1.32) and (3.87) that $B(F)$ is unitarily similar to a compact perturbation of B (see (2.74)) and hence, by Weyl's theorem, the operators $B(F)$ and B have the same essential spectrum, namely (see (2.81)) $\{\beta_k\}_{k=1}^m$. It follows from theorem 2.3 that $\sigma_{\text{ess}}(\mathcal{A}(F)) = \{+\infty\} \cup \{\beta_k\}_{k=1}^m$. Let $\mathcal{L}_-$ be the maximal nonpositive invariant subspace of $\mathcal{A}(F)^{-1}$ and $\mathcal{A}_- := (\mathcal{A}(F)^{-1} \mid \mathcal{L}_-)$. Then $\sigma_{\text{ess}}(\mathcal{A}_-) = \{1/\beta_k\}_{k=1}^m$. Consider a factor space $\widehat{\mathcal{L}}_- := \mathcal{L}_-/\mathcal{L}_-^\circ$, where $\mathcal{L}_-^\circ$ is the linear hull of neutral vectors from $\mathcal{L}_-$; it is finite-dimensional as $\mathcal{L}_- \in \mathbf{h}^-$. A bilinear form $-[\cdot,\cdot]$ naturally generates in $\widehat{\mathcal{L}}_-$ a scalar product with respect to which $\widehat{\mathcal{L}}_-$ is a Hilbert space. Also in a natural way the operator $\mathcal{A}_-$ induces in $\widehat{\mathcal{L}}_-$ a selfadjoint operator $\widehat{\mathcal{A}}_-$ with a finite number of points $\{1/\beta_k\}_{k=1}^m$ of essential spectrum. Therefore these points are either points of accumulation of finite-to-one eigenvalues of $\widehat{\mathcal{A}}_-$ or they are its eigenvalues of infinite multiplicity, or both cases take place at the same time. Furthermore, in $\widehat{\mathcal{L}}_-$ there exists orthonormal basis consisting of eigenvalues of $\widehat{\mathcal{A}}_-$. It follows from finite dimensionality and $\mathcal{A}_-$-invariance of $\mathcal{L}_-^\circ$ that either $\{1/\beta_k\}_{k=1}^m$ are points of accumulation of finite-to-one eigenvalues of $\mathcal{A}_-^{-1}$ or they are its eigenvalues of infinite multiplicity, or both cases take place at the same time. Also, the closed linear hull of eigen vectors corresponding to the positive eigenvalues has a finite defect in $\mathcal{L}_-$. Reasoning as in the proof of theorem 2.7 and replacing the maximal nonnegative invariant subspace $\mathcal{L}_+$ by the maximal nonpositive invariant one $\mathcal{L}_-$, we obtain the proof of assertions of the theorem on the branches of eigenvalues, on completeness with a finite defect of projections of eigenvalues on $\widehat{\mathcal{H}}_1$ and their basicity under condition (3.85).

It remains to check that under condition (3.86) projections of vectors $\{\hat{u}_{1n}^{(j)}\}_{n=1}^\infty$ on $\mathcal{H}_j$, $j = \overline{1,m}$, form a Riesz basis with a finite defect in $\mathcal{H}_j = \mathcal{H}$, $j = \overline{1,m}$. In the sequel, if a misunderstanding cannot arise, we shall omit the argument F at the operator $\mathcal{A}$ and its derivatives.

Taking into account representation (1.40) for the operator $\mathcal{A}$ and the fact that $\hat{u}_1 = (u_1; \ldots; u_m)^t \in \widehat{\mathcal{H}}_1$, we rewrite (2.53) as the following system in components u_k, $k = \overline{1,m}$:

$$A_0(u_0 + \sum_{k=1}^m A_0^{-1/2} Q_k^* u_k) = \lambda u_0, \tag{3.88}$$

$$-Q_k A_0^{1/2} u_0 + \gamma_k u_k = \lambda u_k \quad (k = \overline{1,m}). \tag{3.89}$$

It follows directly from this system that none of numbers $\{\gamma_k\}_{k=1}^m$ is not an eigenvalue of $\mathcal{A}$. Since these numbers are not points of accumulation of the spectrum of $\mathcal{A}$, they are regular. Therefore we can obtain u_k from (3.89) and pass to the system

$$A_0(u_0 + \sum_{k=1}^{m} (\gamma_k - \lambda)^{-1} A_0^{-1/2} Q_k^* Q_k A_0^{1/2} u_0) = \lambda u_0, \tag{3.90}$$

$$u_k = (\gamma_k - \lambda)^{-1} Q_k A_0^{1/2} u_0, \quad k = \overline{1, m}. \tag{3.91}$$

As for elements $\tilde{u} = (u_0; \hat{u}_1)^t \in \mathcal{D}(\mathcal{A})$ the inclusion $u_0 \in \mathcal{D}(\mathcal{A}_0^{1/2})$ is valid, we can realize in (3.90) the change

$$A_0^{1/2} u_0 = v, \quad v \in \mathcal{H}, \tag{3.92}$$

and apply from the left in (3.90) the operator $A_0^{-1/2}$. Then, taking into account (3.82), for obtaining of elements v the following spectral problem arise (see (3.83)):

$$A(\lambda) v = 0. \tag{3.93}$$

Since $A(\beta_j) \in \mathfrak{S}_\infty$, $j = \overline{1, m}$, and requirement (3.86) holds, we are in conditions [28, Corollary 33.7], under which in neighborhoods of the points β_j the operator-valued function $A(\lambda)$ have eigenvalues $\{\lambda_k^{(j)}\}_{k=1}^\infty$ such that the corresponding eigen elements $\{v_k^{(j)}\}_{k=1}^\infty$, $j = \overline{1, m}$, form in $\mathcal{H}_0$ a Riesz basis with a finite defect. It remains to use relations (3.92), (3.91) and take into account that γ_j, $j = \overline{1, m}$, are not points of accumulation of the spectrum of $\mathcal{A}$, and Q_j, $j = \overline{1, m}$, are continuous and continuously invertible operators. ■

3.2 Asymptotic behavior of branches of eigenvalues

On the operator coefficients A_k we impose additional conditions that allow to determine principal terms of asymptotic behavior for branches of eigenvalues of problem (3.93) under the conditions of theorem 3.2. Note that these conditions are typical for operators A_k occurring in problems of mathematical physics.

Theorem 3.3 *Let relations* (3.82), (3.85), (3.86), *and also conditions*

$$\lambda_n(A_0) = a_0^{-1/\alpha_0} n^{1/\alpha_0} [1 + o(1)] \ (n \to \infty), \quad a_0 > 0, \ \alpha_0 > 0, \tag{3.94}$$

$$\lambda_n^{\pm}(\tilde{A}_{0j}) = \pm (a_j^{\pm})^{1/\alpha_j^{\pm}} n^{-1/\alpha_j^{\pm}} [1 + o(1)] \quad (n \to \infty), \tag{3.95}$$

$$a_j^{\pm} > 0, \quad \alpha_j^{\pm} > 0, \quad j = \overline{1, m},$$

for eigenvalues $\lambda_n(A_0)$ of A_0 and positive and negative eigenvalues $\lambda_n^{\pm}(\tilde{A}_{0j})$ of $\tilde{A}_{0j} := -A(\beta_j)$, $j = \overline{1, m}$, are fulfilled.

Then the branch $\{\lambda_n^{(\infty)}\}_{n=1}^{\infty}$ *has the following asymptotic behavior*

$$\lambda_n^{(\infty)} = \lambda_n(A_0)[1 + o(1)] \quad (n \to \infty), \tag{3.96}$$

and each of the branches $\{\lambda_n^{(j)}\}_{n=1}^{\infty}$ *can be divided into two subbranches* $\{\lambda_n^{(j,\pm)}\}_{n=1}^{\infty}$ *located respectively to the right and to the left of the point* β_j, *and having the asymptotic behavior*

$$\lambda_n^{(j,\pm)} = \beta_j + \lambda_n^{\pm}(\tilde{A}_{0j})(e_0'(\beta_j))^{-1}[1 + o(1)] \quad (n \to \infty,\, j = \overline{1,m}). \tag{3.97}$$

Proof is based on using [29, Theorem 3]. Rewrite problem (3.93) in the form

$$[I - \lambda A_0^{-1} + \Psi(\lambda)]v = 0, \quad \Psi(\lambda) := \sum_{k=1}^{m} (\gamma_k - \lambda)^{-1}(\alpha_k I + F_k). \tag{3.98}$$

Since $\Psi(\lambda)$ is holomorphic at ∞ operator-function, $\Psi(\lambda) \to 0$ $(\lambda \to \infty)$, and the operator A_0^{-1}, by (3.94), has a power asymptotic behavior of eigenvalues $\lambda_n(A_0^{-1}) = 1/\lambda_n(A_0)$ as $n \to \infty$, from [29, Theorem 3] (see [30, 31] for details) we obtain formula (3.96).

let us divide the two sides of (3.93) by $e_0(\lambda)$ and replace the spectral parameter by $\lambda = \beta_j + \mu$ for certain j. Instead of (3.93) we get the problem

$$M_j(\mu)v := (-\tilde{A}_{0j} + \mu e_0'(\beta_j)\tilde{A}_j + \tilde{A}_{2j}(\mu))v = 0, \tag{3.99}$$

$$\tilde{A}_{2j}(\mu) := m_{0j}(\mu)I + \sum_{k=1}^{m} F_k \psi_{kj}(\mu) = O(\mu^2) \quad (\mu \to 0),$$

$$m_{0j}(\mu) := e_0(\beta_j + \mu) - e_0'(\beta_j)\mu = O(\mu^2),$$

$$\psi_{kj}(\mu) := (\gamma_k - \beta_j - \mu)^{-1} - (\gamma_k - \beta_j)^{-1} - (\gamma_k - \beta_j)^{-2}\mu = O(\mu^2), \tag{3.100}$$

where $\tilde{A}_j := A'(\beta_j)/e_0'(\beta_j)$ (see (3.84)).

Assume now in (3.99) $\mu = \tilde{\mu}^{-1}$ and multiply the two sides of the equation by $\tilde{\mu}$. On account of (3.86), we have

$$\widetilde{M}_j(\tilde{\mu})v := \left[e_0'(\beta_j)I - \tilde{\mu}\tilde{A}_{0j} - T_{1j} + \tilde{\mu}\tilde{A}_{2j}(\tilde{\mu}^{-1})\right]v = 0, \tag{3.101}$$

where $T_{1j} := A_0^{-1} - \sum_{k=1}^{m}(\beta_j - \gamma_k)^{-2}F_k = T_{1j}^* \in \mathfrak{S}_\infty$, and $\tilde{\mu}\tilde{A}_{2j}(\tilde{\mu}^{-1}) \to 0$ $(\tilde{\mu} \to \infty)$.

Taking into account asymptotic formulas (3.95) and applying again [29, Theorem 3], we conclude that the branch of eigenvalues $\tilde{\mu}_{jn} = 1/\mu_{jn}$ with the limit point at ∞ is divided into two subbranches $\{\tilde{\mu}_{jn}^{+}\}$ and $\{\tilde{\mu}_{jn}^{-}\}$ consisting of positive and negative eigenvalues and having asymptotic behavior

$$\tilde{\mu}_{jn}^{\pm} = e_0'(\beta_j)/\lambda_n^{\pm}(\tilde{A}_{0j})[1 + o(1)] \quad (n \to \infty,\ j = \overline{1,m}),$$

whence, on account of the change $\lambda = \beta_j + \tilde{\mu}_j^{-1}$, formulas (3.97) follow. ■

Note that the character of asymptotic behavior of the branches $\lambda_n^{(j,\pm)}$ is determined not only by properties of A_0, but also by properties of compact operators F_k from (3.82). In particular, it follows from (3.83) that if $F_k \geq 0$, then, by the inequalities $\beta_k - \gamma_k > 0$, operators $\tilde{A}_{0k} = -A(\beta_k) > 0$ and hence branches $\lambda_n^{(j,-)}$ from (3.97) are absent. Besides, the principal term in asymptotic formula (3.97) can be determined not only by A_0^{-1}, but also by F_k.

3.3 A separate basicity of the system of eigenvalues

As was stated in the proof of theorem 3.2, to each branch of eigenvalues $\{\lambda_n^{(j)}\}_{n=1}^{\infty}$ with the limit point β_j there correspond eigen elements of problem (3.93) forming a Riesz basis up to a finite defect in the Hilbert space $\mathcal{H}$. We show that under certain additional conditions on the coefficients F_k $(k = \overline{1,m})$ and A_0^{-1}, the mentioned bases are p-bases. To state this, we need the following result [32].

Theorem 3.4 *Let $L(\lambda) := \lambda I + \sum\limits_{k=0}^{\infty} \lambda^k A_k$ be a holomorphic at zero selfadjoint operator-valued function, and also $A_k \in \mathfrak{S}_{p_k}$, $k = \overline{0,q}$, $q \geq 1$. If the condition $L'(0) = I + A_1 \gg 0$ is fulfilled, then there exists $\varepsilon > 0$ such that the system of eigen elements of $L(\lambda)$ corresponding to eigenvalues from the segment $[-\varepsilon, \varepsilon]$ form p-basis with a finite defect in $\mathcal{H}$ for*

$$p \geq \left[\min(p_1^{-1}, p_0^{-1} + p_2^{-1}, 2p_0^{-1} + p_3^{-1}, \ldots, (q-1)p_0^{-1} + p_q^{-1}, qp_0^{-1})\right]^{-1} =: \tilde{p}. \tag{3.102}$$

Theorem 3.5 *Suppose requirements* (3.82), (3.85), (3.86), *and also the conditions*

$$A_0^{-1} \in \mathfrak{S}_{p_0}, \quad F_k \in \mathfrak{S}_{p_k}, \quad (k = \overline{1,m}) \tag{3.103}$$

are fulfilled. Then in problem (3.93) *the eigen elements $\{v_n^{(j)}\}_{n=1}^{\infty}$ corresponding to the eigenvalues $\{\lambda_n^{(j)}\}_{n=1}^{\infty}$ with the limit point β_j, form p-basis in $\mathcal{H}$ up to a finite defect for*

$$p \geq \widehat{p}_0 := \max(p_0, \max_{1\leq k\leq m} p_k). \tag{3.104}$$

The system of the eigen elements $\{v_n^{(\infty)}\}_{n=1}^{\infty}$ corresponding to the eigenvalues $\{\lambda_n^{(\infty)}\}_{n=1}^{\infty}$ with the limit point $\lambda = \infty$, form p-basis in $\mathcal{H}$ up to a finite defect for $p \geq p_0$.

Proof. First note that the assertion for the branches tending to $+\infty$, in more general assumptions, was proved in theorem 2.7. Therefore we prove only the first part of the conclusion.

Realize in (3.93) the change of the spectral parameter $\lambda = \beta_k + \mu$ and retain all the terms up to the order $O(\mu^2)$ $(\mu \to 0)$. Applying to the two sides of obtained equation the function

$$\psi(\mu) := \left[1 + \frac{\mu e_0''(\beta_k)}{2e_0'(\beta_k)} + \frac{\mu^2 e_0'''(\beta_k)}{6e_0'(\beta_k)} + O(\mu^3)\right]^{-1}, \tag{3.105}$$

we get the equation

$$\widetilde{M}_j(\mu)v = 0,\ \widetilde{M}_j(\mu) = \mu e_0'(\beta_j)I + \Phi_{0j} + \mu\Phi_{1j} + \mu^2\Phi_{2j} + O(\mu^3)\ \ (\mu \to 0), \tag{3.106}$$

$$\left.\begin{aligned}
&e_0'(\beta_j) > 0, \quad \Phi_{0j} := -\tilde{A}_{0j} = -\Big(\beta_j A_0^{-1} + \sum_{k=1}^{m}(\gamma_k - \beta_j)^{-1}F_k\Big),\\
&\Phi_{1j} := \frac{e_0''(\beta_j)}{2e_0'(\beta_j)}\tilde{A}_{0j} - T_{1j}, \quad T_{1j} = A_0^{-1} - \sum_{k=1}^{m}(\gamma_k - \beta_j)^{-2}F_k,\\
&\Phi_{2j} := \left[\frac{e_0'''(\beta_j)}{6e_0'(\beta_j)} - \left(\frac{e_0''(\beta_j)}{2e_0'(\beta_j)}\right)^2\right]\tilde{A}_{0j} + \frac{e_0''(\beta_j)}{2e_0'(\beta_j)}T_{1j} + \sum_{k=1}^{m}(\beta_j - \gamma_k)^{-3}F_k.
\end{aligned}\right\} \tag{3.107}$$

Structure (3.107) of operators Φ_{ij}, $i = \overline{0,2}$, shows that under assumptions (3.103) all of them belongs to the class $\mathfrak{S}_{\widehat{p}_0}$, where $\widehat{p}_0$ is defined by formula (3.104). We apply now theorem 3.4 and note that in this theorem (see [32]) the condition $L'(0) \gg 0$ was used for the proof of the existence of eigenvalue branch with the limit point at zero. In connection with problem (3.106), this condition holds for the operator-valued function $M_j(\mu) = \psi^{-1}(\mu)\widetilde{M}_j(\mu)$, where $\psi(\mu)$ is function (3.105). Hence for problem (3.106) the conclusions of theorem 3.4 are valid for $q = 2$ and numbers $\tilde{p}_0 = \tilde{p}_1 = \tilde{p}_2 = \widehat{p}_0$, which must define the right side of (3.102). It follows from this that this right side equals $\tilde{p} = \widehat{p}_0$, i. e; the first assertion of the theorem, which is being proved. ■

Note that methods of spectral theory of operator pencils allow also to establish (see [28]) sufficient conditions in order that to chosen branches of the spectrum of problem (3.93) there correspond p-bases consisting of eigen vectors without a defect in $\mathcal{H}$.

3.4 Properties of solutions of the problem on normal oscillations in hydrodynamic case

As was noted above, hydrodynamic problem (1.9) on small oscillations of viscoelastic fluid is a special case of general problem (1.1) for $\mathcal{H} = \vec{J}_0(\Omega)$ and changes $A_0 \mapsto \nu A_0$, $A_k \mapsto \nu\alpha_k A_0$, where A_0 from the right is the Stokes operator (subsection 1.2). This important special case for $\nu = 1$ was investigated partially in subsection 2.3. Therefore below properties of solutions of spectral problem (2.53) corresponding to the hydrodynamic situation for any $\nu > 0$ will be formulated without proofs.

First of all, instead of equation (2.79) we have here the problem on eigenvalues

$$\vec{u}_0 = \lambda(\nu e_0(\lambda))^{-1} A_0^{-1} \vec{u}_0, \quad \vec{u}_0 \in \vec{J}_0(\Omega), \quad e_0(\lambda) := 1 + \sum_{k=1}^{m} \alpha_k (\gamma_k - \lambda)^{-1}, \tag{3.108}$$

where A_0 is the Stokes operator with eigenvalues $\lambda_n(A_0)$ having asymptotic behavior (1.13). Characteristic equation for finding of eigenvalues λ (3.108) takes the form

$$\lambda = \nu \lambda_n(A_0) e_0(\lambda), \quad n \in \mathbb{N}. \tag{3.109}$$

Here we mention without proofs properties of solutions of problem (3.109) and formulate physical conclusions for hydrodynamic problem obtained on the base of these properties.

1°. For any $n \in \mathbb{N}$ equation (3.109) has $m+1$ roots; at most two of them can be nonreal complex conjugate, and the rest are positive.

2°. For any $n \in \mathbb{N}$ there exists at most two real roots located to the left of the point $\lambda = \gamma_1$. For sufficiently large n there are no such roots. Thus, there exists at most finite number of roots located to the left of γ_1 for all $n \in \mathbb{N}$. The rest of roots are located to the right of the point $\lambda = \beta_1 > \gamma_1$.

3°. For sufficient large n there are no complex conjugate pairs of nonreal solutions of problem (3.109), hence total number of nonreal eigenvalues in problem (3.108) is at most finite one (this fact was established above). If viscosity ν of fluid is sufficiently large, then problem (3.108) does not have nonreal eigenvalues, and also real eigenvalues located to the left of γ_1.

4°. Eigenvalues of problem (3.108) can be divided into $m+1$ branches with the limit points β_k $(k = \overline{1,m})$ $+\infty$. Furthermore, eigenvalues $\{\lambda_n^{(k)}\}_{n=1}^{\infty}$ corresponding to the branch of number k are located in the interval (β_k, γ_{k+1}), $k = \overline{1,m}$, where γ_{m+1} is multiple root of equation (3.109) with the left side of the form $a\lambda$ for certain $a > 0$, and this root is located to the right of β_m. Eigenvalues $\lambda_n^{(\infty)}$ corresponding to the branch with the limit point $+\infty$ are located in the interval (γ_{m+1}, ∞).

5°. For the branches $\lambda_n^{(k)}$ the following asymptotic formulas hold:

$$\lambda_n^{(k)} = \beta_k + \beta_k (\nu e_0'(\beta_k) \lambda_n(A_0))^{-1} + O(\lambda_n^{-2}(A_0)) \quad (n \to \infty,\ k = \overline{1,m}). \tag{3.110}$$

6°. For the branch $\lambda_n^{(\infty)}$ the following formula takes place as $n \to \infty$

$$\lambda_n^{(\infty)} = \nu \lambda_n(A_0) \left[1 - \left(\sum_{k=1}^{m} \alpha_k \right) \big(\nu \lambda_n(A_0)\big)^{-1} + O\big(\lambda_n^{-2}(A_0)\big) \right]. \tag{3.111}$$

7°. To the branch $\lambda_n^{(\infty)}$ with asymptotic behavior (3.111) there correspond dissipative waves in the visco-elastic fluid related to the usual dissipative waves in the incompressible viscous homogeneous fluid. For these waves damping decrements can be arbitrarily large, and existence of elastic forces in the hydrosystem, as follows from (3.111) at $\alpha_k > 0$, slightly decreases these decrements.

8°. To the branches $\lambda_n^{(k)}$, $k = \overline{1, m}$, there corresponds a new form of normal wave motions caused by the acting in the system visco-elastic forces. Damping decrements corresponding to these waves, i. e; numbers $\lambda_n^{(k)}$, are located in finite intervals of the positive semiaxis and have exactly m limit points (see (3.110)), i. e; their number coincides with the number of integral addends in equations of motion (1.5).

9°. It is possible, by existence of visco-elastic forces, that in the given hydrodynamic system along with aperiodic damping regimes of normal oscillations, which are typical for the usual viscous fluid, oscillating in time (with the frequency $\omega = \operatorname{Im} \lambda \neq 0$) damping (with damping decrement $\operatorname{Re} \lambda$)regimes; there can be at most finite number of such regimes.

Thus, as follows from above, visco-elastic forces generate new physical effects, which are not typical for the usual viscous fluid completely filling an arbitrary container.

Finally we note that recently factorization of operator matrices by Schur-Frobenius (see §1) with the consequent extension of matrix factors used widely in a number of works (see for instance [31 − 37]).

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