Math. Nachr. **248-249**, 3 – 39 (2003) / **DOI** 10.1002/mana.200310001

On the problem of small motions and normal oscillations of a viscous fluid in a partially filled container

Tomas Ya. Azizov[*1], **Volker Hardt**[**2], **Nikolay D. Kopachevsky**[*** 3], and **Reinhard Mennicken**[†4]

[1] The State University of Voronezh, Universitetskaya pl. 1, 394693 Voronezh, Russia
[2] Dr. Hans–Hutter–Straße 1, 85072 Eichstätt, Germany
[3] Mathematical Analysis Chair, Taurida National, V. Vernadsky University, ul. Yaltinskaya, 4, 95007 Simferopol, Crimea, Ukraine
[4] Department of Mathematics, University of Regensburg, Universitätsstr. 31, 93053 Regensburg, Germany

Received 31 May 2000, revised 8 October 2002, accepted 8 October 2002
Published online 8 Januar 2003

Key words Hilbert space, evolution problem, Navier–Stokes equations, normal oscillations
MSC (2000) Primary: 47B25; Secondary: 47A11, 47A56

We dedicate this paper to the memory of Selim Grigor'evich Krein – an outstanding mathematician and man

The famous classical S. Krein problem of small motions and normal oscillations of a viscous fluid in a partially filled container is investigated by a new approach based on a recently developed theory of operator matrices with unbounded entries. The initial boundary value problem is reduced to a Cauchy problem

$$\frac{dy}{dt} + \mathcal{A}y = f(t), \quad y(0) = y^0,$$

in some Hilbert space. The operator matrix $\mathcal{A}$ is a maximal uniformly accretive operator which is selfadjoint in this space with respect to some indefinite metric. The theorem on strong solvability of the hydrodynamic problem is proved. Further, the spectrum of normal oscillations, basis properties of eigenfunctions and other questions are studied.

1 Introduction

1.1 On the history of the problem

We investigate the classical problem on small motions and normal oscillations of a viscous incompressible fluid in an open rigid immovable vessel. By using functional analytical–methods the first investigations of the problem were carried out in the sixties of the last century and are connected with works of S. G. Krein and his students (see [29], [32], [3]). In those publications normal oscillations, i.e., motions with dependence on the time t according to the law $\exp(-\lambda t)$, were studied and the spectral problem

$$L(\lambda)v := \big(I - \lambda A - \lambda^{-1}B\big)v = 0 \tag{1.1}$$

was obtained where A is a compact and positive operator and B a compact and nonnegative operator acting in some Hilbert space.

Problem (1.1) generated numerous investigations. We mention here the works of W. H. Greenlee [15, 16], A. S. Markus and V. I. Matsaev [34], A. G. Garadzhaev [13], N. D. Kopachevsky [20] – [22]. Two approaches were applied for studying the problem (1.1). The first one was based on a direct investigation of the operator

* e–mail: azizov@tom.vsu.ru
** e-mail: volker.hardt@mathematik.uni-regensburg.de, Phone: +49 - (0)8421 - 8420
*** e–mail: kopachevsky@tnu.crimea.ua
† Corresponding author: e–mail: reinhard.mennicken@mathematik.uni-regensburg.de

 0025-584X/03/248-24901-0003 $ 17.50+.50/0

pencil $L(\lambda)$ (factorization methods etc.), the second one was the method of changing the spectral parameter to reduce the problem (1.1) to an eigenvalue problem for some operator matrix with bounded entries. The transformations $\mu = \lambda + \lambda^{-1}$ (S. G. Krein, G. I. Laptev), $\mu = \lambda - \lambda^{-1}$ (W. H. Greenlee) were used, however the obtained spectral problem was in some sense not adequate to the original initial boundary value problem on small motions of a viscous fluid in a vessel. In particular, the solutions of the initial problem cannot be developed into a Fourier series by the corresponding eigensolutions.

The second way of investigations of the problem (1.1) was based on the application of operator theory in spaces with an indefinite metric, in particular, on using deep results of M. G. Krein and H. Langer [27, 28] on quadratic operator pencils. In this direction we mention the results of E. A. Larionov [33], T. Ya. Azizov [5], A. G. Kostjuchenko and A. A. Shkalikov [24, 25], T. Ya. Azizov and N. D. Kopachevsky [7]. The main point in these investigations is the reduction of equations with unbounded operator coefficients to equations with bounded ones. On this way it is however difficult to see the connection between the solutions of the evolution equation and the corresponding spectral problem.

Detailed applications of the above–mentioned approaches and a description of the solution properties of the problem (1.1) and the corresponding initial boundary value problem was considered in the monograph by N. D. Kopachevsky, S. G. Krein and Ngo Zuy Can [23, Chapter 7].

1.2 Application of operator matrix theory with unbounded entries

Recently operator matrices became an effective tool of investigations in some problems of hydrodynamics and magnetohydrodynamics. These matrices act in orthogonal sums of Hilbert spaces and have unbounded entries. The domains of the entries have to be related in some way. As a rule, such operator matrices are a priori not closed. The research of these mathematical objects was carried out in the papers of F. V. Atkinson, H. Langer, R. Mennicken, A. A. Shkalikov [4], V. M. Adamyan, H. Langer [1], V. M. Adamyan, H. Langer, R. Mennicken, J. Saurer [2], M. Fairman, R. Mennicken, M. Möller [9] – [11], V. Hardt, R. Mennicken, S. Naboko [17], R. Mennicken, A. A. Shkalikov [35], A. A. Shkalikov [39], A. Yu. Konstantinov [19], T. Ya. Azizov, N. D. Kopachevsky, L. D. Orlova [8] and others.

As a final result, it was shown that the (closure of the) operator matrices lead to unbounded operators, which are adequate, in some sense, to both the initial boundary value problem and the corresponding spectral problem.

This new approach will be applied in the present paper to the problem on small oscillations of a viscous fluid in an open vessel. First, we reduce the problem to a Cauchy problem

$$\frac{dy}{dt} + \mathcal{A}y \;=\; f\,, \quad y(0) \;=\; y^0\,, \tag{1.2}$$

in the orthogonal sum of two Hilbert spaces, which is connected with the velocity field of the viscous fluid in a vessel and with the vertical displacement field of the moving free surface by the relation (3.11). Therefore, the operator coefficient $\mathcal{A}$ in the equation (1.2) is a 2×2 operator matrix acting in the orthogonal sum of two Hilbert spaces.

The spectral problem

$$\mathcal{A}y \;=\; \lambda y\,, \quad y \in \mathcal{D}(\mathcal{A})\,, \tag{1.3}$$

corresponding to the evolution problem (1.2) is investigated. The theorem on strong solvability of the initial boundary value problem is proved by using the fact that the operator $\mathcal{A}$ is a maximal uniformly accretive and $\mathcal{J}$–selfadjoint operator. The representation of the solution of the initial boundary value problem in a Fourier series by root elements of the spectral problem (1.3) is proved.

2 Preliminary information

In the present paper we use some assertions which arise in functional analysis and the theory of elliptic boundary value problems. Further we apply some statements which stem from the theory of vector–valued Sobolev spaces. These assertions are written down in this section for the convenience of the reader.

2.1 The spaces $E^{\frac{1}{2}}$ and $E^{-\frac{1}{2}}$

The Hilbert space F is called densely embedded in the Hilbert space E if F is a dense linear subset of E and there exists a constant $\alpha > 0$ such that

$$\|x\|_E \;\leq\; a\,\|x\|_F \tag{2.1}$$

for all $x \in F$.

Definition 2.1 The Hilbert spaces F and E form a *Hilbert pair* $(F; E)$ if F is densely embedded in the Hilbert space E.

Let $(F; E)$ be a Hilbert pair and V the adjoint of the embedding of F into E. The operator V is positive as an operator acting from E into E. If F is compactly embedded in the space E, then V is a compact operator in E as the adjoint of a compact operator.

Consider the inverse operator $A := V^{-1} : \mathcal{D}(A) \subseteq F \to E$. The operator A has the following property

$$(x, Az)_E \;=\; (x, z)_F \quad (x \in F,\; z \in \mathcal{D}(A))\,. \tag{2.2}$$

Further the square root of A exists. It is well–known (see, for example, [18] Chap. VI Theorem 2.1 and Theorem 2.23) that $\mathcal{D}\big(A^{\frac{1}{2}}\big) = F$ and

$$\big(A^{\frac{1}{2}}x, A^{\frac{1}{2}}y\big)_E \;=\; \big(x, y\big)_F \quad (x,\, y \in F \subset E)\,. \tag{2.3}$$

Definition 2.2 The operator A is said to be the *generating operator* of the Hilbert pair $(F; E)$.

We introduce the Hilbert space $E^{\frac{1}{2}}$ with the help of the operator A in the usual way (see, for example, [23]). We denote the domain of the operator $A^{\frac{1}{2}}$ by $E^{\frac{1}{2}}$. This vector space is a Hilbert space with scalar product

$$\big(x, y\big)_{E^{\frac{1}{2}}} \;:=\; \big(A^{\frac{1}{2}}x, A^{\frac{1}{2}}y\big)_E \quad \big(x,\, y \in \mathcal{D}\big(A^{\frac{1}{2}}\big) \subset E\big)\,. \tag{2.4}$$

We conclude from formula (2.3) that $F = E^{\frac{1}{2}}$ and $\|x\|_F = \|x\|_{E^{\frac{1}{2}}}$. Further we introduce a new norm on the space E by

$$\|x\|_{E^{-\frac{1}{2}}} \;:=\; \big\|A^{-\frac{1}{2}}x\big\|_E \quad (x \in E)\,. \tag{2.5}$$

$E^{-\frac{1}{2}}$ denotes the completition of the space E with respect to this norm. We remark that the spaces E^{α} and $E^{-\alpha}$ can be introduced in the same way for $\alpha \geq 0$ and the set E^{α} $(-\infty < \alpha < \infty)$ is called a scale of Hilbert spaces.

In the following we describe the space $E^{-\frac{1}{2}}$ more explicitly. For $x \in E$ we define the antilinear functional l_x on F by $l_x(y) := (x, y)_E$ $(y \in F)$. From the inequality

$$\big|l_x(y)\big| \;=\; \big|\big(A^{-\frac{1}{2}}x, A^{\frac{1}{2}}y\big)_E\big| \;\leq\; \big\|A^{-\frac{1}{2}}x\big\|_E \,\big\|A^{\frac{1}{2}}y\big\|_E \;=\; \big\|A^{-\frac{1}{2}}x\big\|_E\, \|y\|_{E^{\frac{1}{2}}}$$

for all $y \in E^{\frac{1}{2}}$ we obtain that l_x is bounded and $\big\|l_x\|_{F^*} \leq \big\|A^{-\frac{1}{2}}x\big\|_E$. For $z = A^{-1}x$ we conclude that

$$l_x(z) \;=\; \big(A^{-\frac{1}{2}}x, A^{-\frac{1}{2}}x\big)_E = \;\big\|A^{-\frac{1}{2}}x\big\|_E\,\big\|A^{\frac{1}{2}}\big(A^{-1}x\big)\big\|_E \;=\; \big\|A^{-\frac{1}{2}}x\big\|_E\,\|z\|_{E^{\frac{1}{2}}}\,.$$

Therefore,

$$\|l_x\|_{F^*} \;=\; \big\|A^{-\frac{1}{2}}x\big\|_E\,.$$

Hence, the space E can be embedded into the space F^* of bounded linear functionals on F. The above formulae yield that the completition $E^{-\frac{1}{2}}$ of the space E can isometrically identifyed with some subspace of F^*. We will show that this subspace coincides with the whole space F^*. Indeed, let $y_0 \in F$ such that $f(y_0) = 0$ for all functionals f from the above mentioned subspace. Then for all $x \in E$ we have $0 = l_x(y_0) = (x, y_0)_E$, and therefore $y_0 = 0$. Consequently, the space $E^{-\frac{1}{2}}$ is isometrically identified with the space F^*, i.e., $E^{-\frac{1}{2}} =$

$F^* = \left(E^{\frac{1}{2}}\right)^*$. Let $\langle\,.\,,\,.\,\rangle_E$ be the canonical sesquilinear form with respect to the pair $\left(E^{-\frac{1}{2}}, E^{\frac{1}{2}}\right) = (F^*, F)$. In particular we have

$$\langle x, y\rangle_E \;=\; l_x(y) \quad \left(x \in E,\, y \in E^{\frac{1}{2}} = F\right). \tag{2.6}$$

The above considerations yield that the formula

$$\langle x, y\rangle_E \;=\; \lim_{n\to\infty} (x_n, y)_E\,, \quad \left(\{x_n\} \subset E,\, x_n \to x \text{ in } E^{-\frac{1}{2}},\, y \in E^{\frac{1}{2}} = F\right), \tag{2.7}$$

and the inequality

$$\left|\langle x, y\rangle_E\right| \;\le\; \|x\|_{E^{-\frac{1}{2}}}\, \|y\|_{E^{\frac{1}{2}}} \quad \left(x \in E^{-\frac{1}{2}} = F^*,\, y \in E^{\frac{1}{2}} = F\right). \tag{2.8}$$

Remark 2.3 If F is not a linear subset of E, but there exists a one–to–one mapping of the space F onto a dense set in E and this mapping conserves algebraic operations (a so called embedding operator of F into E), then one can identify F with the range of this mapping. If the inequality $\|x\|_E \le a\,\|x\|_F$ holds, then all arguments mentioned above are conserved.

2.2 Generalizations

In this section we consider the case that the mapping mentioned in Remark 2.3 is not one–to–one.

Let the linear operator γ (in the examples it is the trace operator) be defined on the Hilbert space F and act boundedly into the Hilbert space G, i.e., $\gamma : F \to G$ such that

$$\|\gamma u\|_G \;\le\; b\,\|u\|_F \quad (u \in F)\,. \tag{2.9}$$

Denote the kernel of γ by N, i.e., $N := \operatorname{Ker}\gamma$. The equation (2.9) yields that N is a subspace (closed linear subset) of the space F and we define

$$M \;:=\; F \ominus N \;=\; N^{\perp}\,. \tag{2.10}$$

Further we denote the range $\mathcal{R}(\gamma)$ of the operator γ by $G_+ \subset G$. It is evident that the restriction

$$\gamma_h \;:=\; \gamma|_M \tag{2.11}$$

(in the examples M is the subspace of holomorphic functions) of the operator γ to the subspace M realizes a one–to–one mapping of M onto G_+. It allows us to introduce the structure of a Hilbert space on G_+ by the rule

$$(\phi, \psi)_{G_+} \;:=\; (u, v)_F \quad (u,\, v \in M,\; \gamma_h u = \phi,\; \gamma_h v = \psi)\,. \tag{2.12}$$

Then

$$\|\phi\|_{G_+} \;=\; \|u\|_F \quad (u \in M,\; \gamma_h u = \phi)\,. \tag{2.13}$$

Let $u \in F$ and $\gamma u = \phi$. For any element $\phi \in G_+$ exists a unique element $v \in M$ such that $\gamma_h v = \phi$. Obviously $u - v \in N$. Further,

$$\|u\|_F^2 \;=\; \|v + (u - v)\|_F^2 \;=\; \|v\|_F^2 + \|u - v\|_F^2\,.$$

Hence, the equation (2.13) and this equality yield that

$$\|\phi\|_{G_+}^2 \;=\; \|v\|_F^2 \;\le\; \|u\|_F^2 \quad (u \in F,\; \gamma u = \phi)\,.$$

Therefore

$$\|\phi\|_{G_+} \;=\; \min_{\gamma u = \phi} \|u\|_F\,. \tag{2.14}$$

Suppose now that G_+ is densely embedded into G. It follows from the inequality (2.9) that for $\phi \in G_+$, $u \in M$ and $\gamma_h u = \phi$

$$\|\phi\|_G \;=\; \|\gamma_h u\|_G \;\le\; b\,\|u\|_F \;=\; b\,\|\phi\|_{G_+}\,. \tag{2.15}$$

This means that the spaces G_+ and G form a Hilbert pair $(G_+;G)$. Therefore, we can construct the Hilbert spaces $G^{\pm\frac{1}{2}}$ such that $G_+ = G^{\frac{1}{2}}$ and $G^{-\frac{1}{2}}$ can be identified with the space $(G_+)^*$.

Consider the operator γ_h as a mapping from the Hilbert space $\big(M,(\,.\,,\,.\,)_F\big)$ into the Hilbert space $\big(G_+,(\,.\,,\,.\,)_{G_+}\big)$. Let γ_h^* denote its adjoint operator. Since the operator γ_h maps the space M isometrically onto the space $G_+ = G^{\frac{1}{2}}$ (see (2.13)), i.e.,

$$\|\gamma_h u\|_{G_+} \;=\; \|u\|_F \quad (u \in M)\,, \tag{2.16}$$

the operator γ_h^* maps the space G_+ isometrically onto the space M. Identifying the isometrically isomorphic spaces G_+ and $G^{-\frac{1}{2}} = (G_+)^*$, we consider γ_h^* as an isometric operator from $G^{-\frac{1}{2}}$ onto M. By definition

$$\big(\gamma_h^*\psi, v\big)_F \;=\; \big(\psi, \gamma_h v\big)_{G_+} \;=\; \langle \psi, \gamma_h v\rangle_G \quad \big(\psi \in G^{-\frac{1}{2}},\, v \in M\big)\,. \tag{2.17}$$

We introduce the operator

$$\partial_h \;:=\; (\gamma_h^*)^{-1} \;:\; M \;\longrightarrow\; G^{-\frac{1}{2}}\,. \tag{2.18}$$

Then the equation (2.17) yields that

$$(u,v)_F \;=\; \langle \partial_h u, \gamma_h v\rangle_G \quad (u,v \in M)\,. \tag{2.19}$$

It follows from the properties of the operators γ_h and γ_h^* that the operator $\widetilde{C} := \gamma_h\gamma_h^*$ maps the space $G^{-\frac{1}{2}}$ isometrically onto the space $G^{\frac{1}{2}} \subset G$. The restriction $C := \widetilde{C}|_G$ is a bounded operator acting in the space G. To prove the last assertion, let $\phi \in G$ and $v \in M$. We conclude from the equations (2.17) and (2.9) that

$$\big|\big(\gamma_h^*\phi, v\big)_F\big| \;=\; \big|\langle \phi, \gamma_h v\rangle_G\big| \;=\; \big|(\phi, \gamma_h v)_G\big| \;\le\; \|\phi\|_G\,\|\gamma_h v\|_G \;\le\; b\,\|\phi\|_G\,\|v\|_F\,.$$

Substituting $v = \gamma_h^*\phi$ we obtain

$$\|\gamma_h^*\phi\|_F^2 \;\le\; b\,\|\phi\|_G\,\|\gamma_h^*\phi\|_F\,,$$

and therefore

$$\|\gamma_h^*\phi\|_F \;\le\; b\,\|\phi\|_G \quad (\phi \in G)\,. \tag{2.20}$$

Consequently, by (2.9), (2.20)

$$\|C\phi\|_G \;=\; \|\gamma_h\gamma_h^*\phi\|_G \;\le\; b\,\|\gamma_h^*\phi\|_F \;\le\; b^2\,\|\phi\|_G \quad (\phi \in G)\,, \tag{2.21}$$

and therefore the operator C is bounded.

If the embedding of the space G_+ into G is compact, i.e., the (trace) operator γ is compact, then the operator C is compact on the space G.

It follows from the identity (2.17) that

$$\big(\gamma_h^*\psi, \gamma_h^*\phi\big)_F \;=\; \langle \psi, \gamma_h\gamma_h^*\phi\rangle_G \;=\; \big(\psi, \gamma_h\gamma_h^*\phi\big)_G \;=\; (\psi, C\phi)_G \tag{2.22}$$

for $\phi, \psi \in G$. Therefore, the operator C is selfadjoint and positive. The formulae (2.22) and (2.12) yield that

$$(C\psi, C\phi)_{G_+} \;=\; (\psi, C\phi)_G \quad (\phi,\, \psi \in G)\,. \tag{2.23}$$

By the substitutions $C\psi = \widetilde{\psi}$, $C\phi = \widetilde{\phi}$ we obtain from (2.23) that

$$\big(\widetilde{\psi}, \widetilde{\phi}\big)_{G_+} \;=\; \big(C^{-1}\widetilde{\psi}, \widetilde{\phi}\big)_G \quad \big(\widetilde{\phi},\, \widetilde{\psi} \in \mathcal{D}\big(C^{-1}\big)\big)\,. \tag{2.24}$$

This identity holds also for $\widetilde{\phi} \in G_+ \supset \mathcal{D}\big(C^{-1}\big)$ which means that the operator C^{-1} is a generative operator of the Hilbert pair $(G_+;G)$ (see (2.2)).

2.3 The main example

Apply the above approach to some special case which is the situation of the problem discussed in this paper.

Consider a region $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ and suppose that $\partial\Omega$ is the union of two surfaces $\partial\Omega := \Gamma \cup \overline{S}$ where $\Gamma \cap \overline{S} = \emptyset$, Γ and S have nonzero surface measure and are open subsets.

Let $H^1(\Omega)$ denote the Sobolev space of first order and introduce the Hilbert space

$$F \ := \ H^1_\Gamma(\Omega) \ := \ \Big\{ u \in H^1(\Omega) \ : \ \int_\Gamma u \, d\Gamma = 0 \Big\}, \tag{2.25}$$

with the norm

$$\|u\|_F \ = \ \|u\|_{H^1_\Gamma(\Omega)} \ := \ \left(\int_\Omega |\nabla u|^2 \, d\Omega \right)^{\frac{1}{2}} . \tag{2.26}$$

$H^1_\Gamma(\Omega)$ is a subspace (with codimension 1) of the usual Sobolev space $H^1(\Omega)$ and the Dirichlet norm (2.26) is equivalent to the standard norm of $H^1(\Omega)$.

We introduce the space $H := L_2(\Gamma) \ominus \{1_\Gamma\}$ of functions which are orthogonal to the identical function 1_Γ defined on Γ in the space $L_2(\Gamma)$. Consider the trace operator

$$\gamma \ : \ H^1_\Gamma(\Omega) \ \longrightarrow \ H \, , \quad \gamma u \ := \ u|_\Gamma \quad \big(u \in H^1_\Gamma(\Omega) \big) \, . \tag{2.27}$$

Its kernel

$$N \ := \ \mathrm{Ker}\,\gamma \ = \ \big\{ u \in H^1_\Gamma(\Omega) : u \equiv 0 \ (\text{on } \Gamma) \big\} \ =: \ H^1_{0,\Gamma}(\Omega) \tag{2.28}$$

is a subspace of the space $H^1_\Gamma(\Omega)$. We consider the subspace

$$M \ := \ F \ominus N \ = \ H^1_\Gamma(\Omega) \ominus H^1_{0,\Gamma}(\Omega)$$

and find an analytic description of the functions from M. To do this, let $v \in M$, $u \in N$ and suppose that v is twice differentiable in $\overline{\Omega}$. The first Green's formula for the Laplace operator yields that

$$0 \ = \ \int_\Omega \nabla v \cdot \overline{\nabla u} \, d\Omega \ = \ \int_\Omega (-\Delta v) \overline{u} \, d\Omega + \int_S \frac{\partial v}{\partial \mathbf{n}} \overline{u} \, dS \tag{2.29}$$

where $\mathbf{n}$ is the unit vector of the external normal to the boundary $\partial\Omega$. We define the space $\overset{\circ}{H}{}^1_\Gamma(\Omega)$ by

$$\overset{\circ}{H}{}^1_\Gamma(\Omega) \ := \ \big\{ w \in H^1_\Gamma(\Omega) \ : \ w = 0 \ (\text{on } \partial\Omega) \big\} \ \subset \ H^1_{0,\Gamma}(\Omega) \tag{2.30}$$

and conclude from formula (2.29) that

$$\int_\Omega (-\Delta v) \overline{u} \, d\Omega \ = \ 0 \quad \Big(u \in \overset{\circ}{H}{}^1_\Gamma(\Omega) \Big) \, . \tag{2.31}$$

Since $\overset{\circ}{H}{}^1_\Gamma(\Omega)$ is dense in the space $L_2(\Omega)$ we obtain that

$$\Delta v \ = \ 0 \quad (\text{in } \Omega) \, . \tag{2.32}$$

The identity (2.29) and the relation (2.32) yield that

$$\int_S \frac{\partial v}{\partial \mathbf{n}} \overline{u} \, dS \ = \ 0 \quad \big(u \in H^1_{0,\Gamma}(\Omega) \big) \, . \tag{2.33}$$

Since the set $\big\{ u|_S \ : \ u \in H^1_{0,\Gamma}(\Omega) \big\}$ is dense in $L_2(S)$ we conclude from (2.33) that

$$\frac{\partial v}{\partial \mathbf{n}} \ = \ 0 \quad (\text{on } S) \, . \tag{2.34}$$

Consequently, the subspace

$$M \;=\; H^1_\Gamma(\Omega) \ominus H^1_{0,\Gamma}(\Omega) \;=:\; H^1_{h,S}(\Omega) \tag{2.35}$$

consists of the following set of (generalized) solutions of the boundary value problems for harmonic functions:

$$H^1_{h,S}(\Omega) \;=\; \left\{ v \in H^1_\Gamma(\Omega) \,:\, \Delta v = 0 \text{ (in } \Omega),\, \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ (on } S) \right\}. \tag{2.36}$$

Thus we have the orthogonal decomposition

$$(F \;=)\; H^1_\Gamma(\Omega) \;=\; H^1_{0,\Gamma}(\Omega) \oplus H^1_{h,S}(\Omega) \;(=\; N \oplus M)\,. \tag{2.37}$$

We introduce the operator

$$\gamma_h \;:=\; \gamma|_M \;=\; \gamma|_{H^1_{h,S}(\Omega)}\,. \tag{2.38}$$

According to Subsection 2.2 γ_h isometrically maps the subspace $M = H^1_{h,S}(\Omega)$ onto the space $H_+ = H_\Gamma^{\frac{1}{2}}$. The norm in $H_\Gamma^{\frac{1}{2}}$ is defined by (see (2.14), (2.16)):

$$\|\phi\|_{H_\Gamma^{\frac{1}{2}}} \;:=\; \min_{\gamma u = \phi} \|u\|_{H^1_\Gamma(\Omega)} \;=\; \|v\|_{H^1_\Gamma(\Omega)} \quad \big(\phi = \gamma_h v,\; v \in H^1_{h,S}(\Omega)\big)\,. \tag{2.39}$$

As a corollary of this fact we obtain the following lemma.

Lemma 2.4 *Let $\phi \in L_2(\Gamma)$. The Zaremba problem*

$$\Delta p \;=\; 0 \quad (\textit{in}\,\Omega)\,, \quad \frac{\partial p}{\partial \mathbf{n}} \;=\; 0 \quad (\textit{on}\, S)\,, \quad \gamma_h p \;=\; \phi \quad (\textit{on}\,\Gamma)\,, \quad \int_\Gamma \phi\, d\Gamma \;=\; 0 \tag{2.40}$$

has a unique (generalized) solution $p \in H^1_{h,S}(\Omega)$ if and only if $\phi \in H_\Gamma^{\frac{1}{2}}$. In this case the solution p fulfills the equality

$$\|p\|_{H^1_\Gamma(\Omega)} \;=\; \|\phi\|_{H_\Gamma^{\frac{1}{2}}}\,. \tag{2.41}$$

Now we consider the operator γ_h^* and its inverse $\partial_h := \left(\gamma_h^*\right)^{-1}$. By definition (see (2.19))

$$(u,v)_{H^1_\Gamma(\Omega)} \;=\; \int_\Omega \nabla u \cdot \overline{\nabla v}\, d\Omega \;=\; \langle \partial_h u, \gamma_h v\rangle_H \quad \big(u,\, v \in H^1_{h,S}(\Omega)\big)\,. \tag{2.42}$$

For twice differentiable functions $u \in H^1_{h,S}(\Omega)$ we conclude from Green's formula and (2.42) that

$$\begin{aligned} \int_\Omega \nabla u \cdot \overline{\nabla v}\, d\Omega \;&=\; \int_\Omega (-\Delta u)\overline{v}\, d\Omega + \int_\Gamma \frac{\partial u}{\partial \mathbf{n}}\,\overline{v}\, d\Gamma + \int_S \frac{\partial u}{\partial \mathbf{n}}\,\overline{v}\, dS \;=\; \int_\Gamma \frac{\partial u}{\partial \mathbf{n}}\,\overline{v}\, d\Gamma \\ &=\; \langle \partial_h u, \gamma_h v\rangle_H \;=\; (\partial_h u, \gamma_h v)_H \;=\; \int_\Gamma \partial_h u \overline{v}\, d\Gamma\,. \end{aligned}$$

Hence,

$$\int_\Gamma \left(\partial_h u - \frac{\partial u}{\partial \mathbf{n}} \right) \overline{v}\, d\Gamma \;=\; 0 \quad \big(v \in H^1_{h,S}(\Omega)\big)$$

and, therefore,

$$\partial_h u \;=\; \left.\frac{\partial u}{\partial \mathbf{n}}\right|_\Gamma . \tag{2.43}$$

The isometry property of the operator

$$\partial_h \;=\; \left(\gamma_h^*\right)^{-1} \;:\; M \;=\; H^1_{h,S}(\Omega) \;\longrightarrow\; H_- \;:=\; (H_+)^* \;=\; \left(H_\Gamma^{\frac{1}{2}}\right)^* \;=:\; H_\Gamma^{-\frac{1}{2}}$$

(see Subsection 2.2) yields the following assertion.

Lemma 2.5 *Let $\psi \in L_2(\Gamma)$. The Neumann problem*

$$\Delta u \;=\; 0 \quad (in\ \Omega)\,, \qquad \frac{\partial u}{\partial \mathbf{n}} \;=\; 0 \quad (on\ S)\,, \qquad \frac{\partial u}{\partial \mathbf{n}} \;=\; \psi \quad (on\ \Gamma)\,, \qquad \int_\Gamma u\, d\Gamma \;=\; 0 \tag{2.44}$$

has a unique (generalized) solution $u \in H^1_{h,S}(\Omega)$ if and only if $\psi \in H_\Gamma^{-\frac{1}{2}} = \left(H_\Gamma^{\frac{1}{2}}\right)^*$. In this case the solution u fulfills the equality

$$\|u\|_{H^1_\Gamma(\Omega)} \;=\; \|\psi\|_{H_\Gamma^{-\frac{1}{2}}}\,. \tag{2.45}$$

2.4 Orthogonal Weyl decomposition of the Hilbert space $\mathbf{L_2(\Omega)}$

In the investigation of the motions of a fluid in an arbitrary region $\Omega \subset \mathbb{R}^3$ we deal with the velocity (or displacement) fields of the moving fluid. These fields must give finite kinetic energy and must therefore belong to the Hilbert space $\mathbf{L}_2(\Omega)$ of vector fields $\mathbf{v}$ with norm

$$\|\mathbf{v}\|^2 \;:=\; \int_\Omega |\mathbf{v}|^2\, d\Omega \tag{2.46}$$

and corresponding scalar product

$$(\mathbf{u},\mathbf{v}) \;:=\; \int_\Omega \mathbf{u}\cdot\overline{\mathbf{v}}\, d\Omega\,. \tag{2.47}$$

Definition 2.6 A field $\mathbf{u} \in \mathbf{L}_2(\Omega)$ is said to be a *potential field* if there exists a $p \in H^1(\Omega)$ such that $\mathbf{u} = \nabla p$ and $\int_\Gamma p\, d\Gamma = 0$ where $\Gamma \subset \partial\Omega$ is a part of $\partial\Omega$ with nonzero surface measure.

For a potential field $\mathbf{u} = \nabla p$ we have

$$\|\mathbf{u}\|^2 \;=\; \int_\Omega |\nabla p|^2\, d\Omega \;<\; \infty\,, \qquad \int_\Gamma p\, d\Gamma \;=\; 0\,. \tag{2.48}$$

In $H^1(\Omega)$ a norm is defined by

$$\|p\|^2_{H^1(\Omega)} \;:=\; \int_\Omega |\nabla p|^2\, d\Omega + \left(\int_\Gamma p\, d\Gamma\right)^2 \qquad \big(p \in H^1(\Omega)\big)$$

which is equivalent to the Dirichlet norm (2.26). If $p \in H^1(\Omega)$, then $u = \nabla p$ is a potential field if and only if

$$\|\nabla p\|^2 \;=\; \|p\|^2_{H^1(\Omega)}\,. \tag{2.49}$$

Remark 2.7 Let $\mathbf{G}(\Omega)$ be the set of all potential fields in $\mathbf{L}_2(\Omega)$. By definition there is a one–to–one correspondence between $\mathbf{G}(\Omega)$ and $H^1_\Gamma(\Omega)$ which is an isometry by (2.49).

Since $H^1_\Gamma(\Omega)$ is a complete space, $\mathbf{G}(\Omega)$ is a subspace of $\mathbf{L}_2(\Omega)$. Denote the orthogonal complement of $\mathbf{G}(\Omega)$ in $\mathbf{L}_2(\Omega)$ by $\mathbf{J}_0(\Omega)$, i.e.,

$$\mathbf{L}_2(\Omega) \;=\; \mathbf{G}(\Omega) \oplus \mathbf{J}_0(\Omega)\,. \tag{2.50}$$

Set $v_\mathbf{n} := \mathbf{v}\cdot\mathbf{n}$ for $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and

$$\hat{\mathbf{J}}_0(\Omega) \;:=\; \left\{\mathbf{u} \in \mathbf{H}^1(\Omega) \;:\; \operatorname{div}\mathbf{u} = 0\ (\text{in}\ \Omega),\ u_\mathbf{n} = 0\ (\text{on}\ \partial\Omega)\right\}. \tag{2.51}$$

We will show that $\mathbf{J}_0(\Omega)$ is the closure of $\hat{\mathbf{J}}_0(\Omega)$: For any smooth field $\mathbf{v} \in \mathbf{J}_0(\Omega)$ and for arbitrary $\nabla p \in \mathbf{G}(\Omega)$ we have

$$0 \;=\; \int_\Omega \nabla p\cdot\overline{\mathbf{v}}\, d\Omega \;=\; \int_\Omega \operatorname{div}\left(p\overline{\mathbf{v}}\right) d\Omega - \int_\Omega p \operatorname{div}\overline{\mathbf{v}}\, d\Omega \;=\; \int_{\partial\Omega} p\,\overline{v_\mathbf{n}}\, dS - \int_\Omega p \operatorname{div}\overline{\mathbf{v}}\, d\Omega\,.$$

For test functions $p \in C_0^\infty(\Omega)$ we obtain that

$$\int_\Omega p \operatorname{div} \overline{\mathbf{v}} \, d\Omega \;=\; 0$$

and therefore $\operatorname{div} \mathbf{v} \;=\; 0$ (in Ω). It follows that

$$\int_{\partial\Omega} p \, \overline{v_{\mathbf{n}}} \, dS \;=\; 0 \quad \big(p \in H^1_\Gamma(\Omega)\big) \tag{2.52}$$

and hence

$$\mathbf{v} \cdot \mathbf{n} \;=\; v_{\mathbf{n}} \;=\; 0 \quad (\text{on } \partial\Omega)\,.$$

Thus, $\mathbf{J}_0(\Omega)$ is contained in the closure of $\hat{\mathbf{J}}_0(\Omega)$. The inverse inclusion is immediate.

We consider a further decomposition of the subspace $\mathbf{G}(\Omega)$. The set $\overset{\circ}{H}{}^1_\Gamma(\Omega)$ (see (2.30)) is a subspace of the space $H^1_\Gamma(\Omega)$ (see Subsection 2.3). Therefore, by Remark 2.7, the set

$$\mathbf{G}_0(\Omega) \;:=\; \Big\{ \mathbf{v} = \nabla u \in \mathbf{L}_2(\Omega) \;:\; u \in \overset{\circ}{H}{}^1(\Omega) \Big\} \tag{2.53}$$

is a subspace of $\mathbf{G}(\Omega)$. Repeating the above mentioned calculations (from (2.29) to (2.32)) we conclude that the orthogonal complement

$$H^1_h(\Omega) \;:=\; H^1_\Gamma(\Omega) \ominus \overset{\circ}{H}{}^1(\Omega)$$

is the space

$$H^1_h(\Omega) \;=\; \big\{ u \in H^1_\Gamma(\Omega) \;:\; \Delta u = 0 \text{ (in } \Omega) \big\}\,. \tag{2.54}$$

Hence we have the decomposition

$$\mathbf{G}(\Omega) \;=\; \mathbf{G}_0(\Omega) \oplus \mathbf{G}_h(\Omega)\,, \tag{2.55}$$

where

$$\mathbf{G}_h(\Omega) \;:=\; \big\{ \mathbf{w} = \nabla\phi \in \mathbf{L}_2(\Omega) \;:\; \phi \in H^1_h(\Omega) \big\}\,. \tag{2.56}$$

As a final result we obtain the so–called Weyl decomposition (see, for instance, [23])

$$\mathbf{L}_2(\Omega) \;=\; \mathbf{J}_0(\Omega) \oplus \mathbf{G}_h(\Omega) \oplus \mathbf{G}_0(\Omega)\,. \tag{2.57}$$

2.5 Operations $\operatorname{div} \mathbf{u}$ and $u_{\mathbf{n}}$ for vector fields from $\mathbf{L}_2(\Omega)$

Let $p \in \overset{\circ}{H}{}^1_\Gamma(\Omega)$, i.e., $\nabla p \in \mathbf{G}_0(\Omega)$, and $\mathbf{u}$ is a smooth field. Then

$$\begin{aligned}(\nabla p, \mathbf{u}) \;=\; \int_\Omega \nabla p \cdot \overline{\mathbf{u}} \, d\Omega \;=\; \int_\Omega \operatorname{div}(p\overline{\mathbf{u}}) \, d\Omega - \int_\Omega p \operatorname{div} \overline{\mathbf{u}} \, d\Omega \;&=\; -\int_\Omega p \operatorname{div} \overline{\mathbf{u}} \, d\Omega \\ &=: \; -\big(p, \operatorname{div} \mathbf{u}\big)_{L_2(\Omega)}\,.\end{aligned} \tag{2.58}$$

If $\mathbf{u} \in \mathbf{L}_2(\Omega)$ is arbitrary and $\mathbf{u}_k \to \mathbf{u}$ $(k \to \infty)$, where the $\mathbf{u}_k$ are smooth, then formula (2.58) yields that

$$(\nabla p, \mathbf{u}) \;=\; -\lim_{k\to\infty} \big(p, \operatorname{div} \mathbf{u}_k\big)_{L_2(\Omega)} \;=:\; -\langle p, \operatorname{div} \mathbf{u}\rangle_{L_2(\Omega)} \tag{2.59}$$

for all $p \in \overset{\circ}{H}{}^1_\Gamma(\Omega)$ and $\mathbf{u} \in \mathbf{L}_2(\Omega)$. Therefore $\operatorname{div} \mathbf{u}$ is a linear bounded functional (distribution) on the space $\overset{\circ}{H}{}^1_\Gamma(\Omega) \subset L_2(\Omega)$ for all $\mathbf{u} \in \mathbf{L}_2(\Omega)$:

$$\operatorname{div} \mathbf{u} \in \Big(\overset{\circ}{H}{}^1_\Gamma(\Omega)\Big)^* \;=:\; H^{-1}(\Omega)\,, \quad \|\operatorname{div} \mathbf{u}\|_{H^{-1}(\Omega)} \;\le\; \|\mathbf{u}\|\,. \tag{2.60}$$

Consider now the operation $u_{\mathbf{n}} := \mathbf{u}\cdot\mathbf{n}$ (on $\partial\Omega$). Let $p \in H^1_\Gamma(\Omega)$, i.e., $\nabla p \in \mathbf{G}(\Omega)$, and let $\mathbf{u} \in \mathbf{L}_2(\Omega)$ be a smooth function and $\operatorname{div}\mathbf{u} = 0$. As above (see (2.58)) we have

$$(\nabla p, \mathbf{u}) \;=\; \int_\Omega \operatorname{div}(p\overline{\mathbf{u}})\, d\Omega - \int_\Omega p \operatorname{div}\overline{\mathbf{u}}\, d\Omega \;=\; -(\gamma p, u_{\mathbf{n}})_{L_2(\partial\Omega)}\,, \tag{2.61}$$

where $\gamma p := p|_{\partial\Omega}$. After limit transition ($\mathbf{u}_k \to \mathbf{u}$ as $k \to \infty$, $\operatorname{div}\mathbf{u}_k = 0$) identity (2.61) yields

$$\big(\nabla p, \mathbf{u}\big) \;=\; -\lim_{k\to\infty}\big(\gamma p, (u_k)_{\mathbf{n}}\big)_{L_2(\partial\Omega)} \;=:\; -\langle \gamma p, u_{\mathbf{n}}\rangle_{L_2(\partial\Omega)} \tag{2.62}$$

for all $p \in H^1_\Gamma(\Omega)$ and $\mathbf{u} \in \mathbf{L}_2(\Omega)$ with $\operatorname{div}\mathbf{u} = 0$. We also use the identity

$$\langle 1_{\partial\Omega}, u_n\rangle_{L_2(\partial\Omega)} \;=\; 0 \tag{2.63}$$

which can be derived as above for $p \equiv 1$. The formulae (2.62) and (2.63) yield that

$$(\nabla p, \mathbf{u}) \;=\; -\,\langle \gamma p, u_{\mathbf{n}}\rangle_{L_2(\partial\Omega)} \quad (p \in H^1(\Omega),\; \mathbf{u} \in \mathbf{L}_2(\Omega),\; \operatorname{div}\mathbf{u} = 0)\,. \tag{2.64}$$

Here the set $\{\gamma p\}$ coincides (by embedding and trace theorems) with the whole space $H^{\frac12}(\partial\Omega)$ if p pass through the space $H^1(\Omega)$. Therefore $u_{\mathbf{n}}$ is a bounded linear functional on the space $H^{\frac12}(\partial\Omega)$ with respect to the scalar product in $L_2(\partial\Omega)$ and

$$u_{\mathbf{n}} \in H^{-\frac12}(\partial\Omega) \;:=\; \big(H^{\frac12}(\partial\Omega)\big)^*, \quad \|u_{\mathbf{n}}\|_{H^{-\frac12}(\partial\Omega)} \;\le\; \|\mathbf{u}\|\,. \tag{2.65}$$

2.6 Orthogonal decomposition of the space $\mathbf{L}_2(\Omega)$ connected with the problem on small oscillations of a fluid in an open vessel

If a homogeneous ideal or viscous fluid partially fills a vessel $\Omega \subset \mathbb{R}^3$ with rigid wall S and free surface Γ then the velocity field of the fluid must be divergence free and the normal component of the field must be equal to zero on S. Since such fields belong to both subspaces $\mathbf{J}_0(\Omega)$ and $\mathbf{G}_h(\Omega)$ in decomposition (2.57), we need an additional decomposition of the subspace $\mathbf{G}_h(\Omega)$. We formulate the corresponding facts without proofs (see, for instance, [23]).

Introduce the subspaces

$$\mathbf{G}_{0,\Gamma,h}(\Omega) \;:=\; \big\{\mathbf{u} = \nabla p \in \mathbf{G}_h(\Omega) \,:\, p = 0 \text{ (on } \Gamma)\big\} \tag{2.66}$$

and

$$\mathbf{G}_{h,S}(\Omega) \;:=\; \left\{\mathbf{u} = \nabla\phi \in \mathbf{G}_h(\Omega) \,:\, u_{\mathbf{n}} = \mathbf{u}\cdot\mathbf{n} = \nabla\phi\cdot\mathbf{n} = \frac{\partial\phi}{\partial\mathbf{n}} = 0 \text{ (on } S)\right\} \tag{2.67}$$

of the subspace $\mathbf{G}_h(\Omega)$. One can check directly that $\mathbf{G}_{0,\Gamma,h}(\Omega)$ and $\mathbf{G}_{h,S}(\Omega)$ are orthogonal. Moreover, the decomposition

$$\mathbf{G}_h(\Omega) \;=\; \mathbf{G}_{0,\Gamma,h}(\Omega) \oplus \mathbf{G}_{h,S}(\Omega) \tag{2.68}$$

holds. Hence, the new decomposition of the space $\mathbf{L}_2(\Omega)$ has the form

$$\mathbf{L}_2(\Omega) \;=\; \mathbf{J}_0(\Omega) \oplus \mathbf{G}_{h,S}(\Omega) \oplus \mathbf{G}_{0,\Gamma,h}(\Omega) \oplus \mathbf{G}_0(\Omega)\,. \tag{2.69}$$

We introduce the subspaces

$$\mathbf{J}_{0,S}(\Omega) \;:=\; \mathbf{J}_0(\Omega) \oplus \mathbf{G}_{h,S}(\Omega)\,, \tag{2.70}$$

$$\mathbf{G}_{0,\Gamma}(\Omega) \;:=\; \mathbf{G}_{0,\Gamma,h}(\Omega) \oplus \mathbf{G}_0(\Omega)\,, \tag{2.71}$$

$$\mathbf{G}(\Omega) \;:=\; \mathbf{G}_{h,S}(\Omega) \oplus \mathbf{G}_{0,\Gamma}(\Omega) \tag{2.72}$$

and obtain

$$\mathbf{J}_0(\Omega) \;=\; \big\{\mathbf{u} \in \mathbf{L}_2(\Omega) \,:\, \operatorname{div}\mathbf{u} = 0 \text{ (in } \Omega),\; u_{\mathbf{n}} = 0 \text{ (on } \partial\Omega)\big\}\,, \tag{2.73}$$

$$\mathbf{J}_{0,S}(\Omega) \;=\; \big\{\mathbf{v} \in \mathbf{L}_2(\Omega) \,:\, \operatorname{div}\mathbf{v} = 0 \text{ (in } \Omega),\; v_{\mathbf{n}} = 0 \text{ (on } S)\big\}\,, \tag{2.74}$$

$$\mathbf{G}_{0,\Gamma}(\Omega) \;=\; \big\{\mathbf{w} = \nabla p \in \mathbf{L}_2(\Omega) \,:\, p = 0 \text{ (on } \Gamma)\big\}\,, \tag{2.75}$$

$$\mathbf{G}(\Omega) \;=\; \left\{\mathbf{v} \in \mathbf{L}_2(\Omega) :\; \mathbf{v} = \nabla p \text{ (in } \Omega),\; \int_\Gamma p\, d\Gamma = 0\right\}. \tag{2.76}$$

Here $\mathbf{J}_{0,S}(\Omega)$ is the subspace of velocity fields of a homogeneous fluid in an open vessel, and the orthogonal decomposition

$$\mathbf{L}_2(\Omega) \;=\; \mathbf{J}_{0,S}(\Omega) \oplus \mathbf{G}_{0,\Gamma}(\Omega) \tag{2.77}$$

holds. In the formulae (2.73) and (2.74) the operators $\operatorname{div}\mathbf{u}$ and $u_{\mathbf{n}}$ are understood as distributions in the sense described in Subsection 2.5.

3 The statement and investigation of the initial boundary value problem

3.1 The statement of the problem

We assume that a rigid immovable vessel is partially filled with a heavy viscous incompressible homogeneous fluid of density $\rho > 0$ and kinematic viscosity $\nu > 0$. The fluid is in an equilibrium state under a gravitational field with constant acceleration $\mathbf{g} = -g\mathbf{e}_3$, where $g > 0$ and $\mathbf{e}_3$ is the unit vector of the vertical axis Ox_3, which is directed opposite to $\mathbf{g}$. $\Omega \subset \mathbb{R}^3$ denotes the domain filled with a fluid in equilibrium state, S the rigid wall of the vessel adherent to the fluid, and Γ the free surface of the fluid. We take the origin O of the Cartesian coordinate system $Ox_1x_2x_3$ on Γ. Then the equation of the surface Γ has the form $x_3 = 0$.

$P_0 = P_0(x)$ denotes the pressure in the fluid in equilibrium state, where $x = (x_1, x_2, x_3) \in \Omega$. Since the equilibrium condition

$$-\rho^{-1}\nabla P_0 \;=\; g\mathbf{e}_3 \tag{3.1}$$

must be fulfilled

$$P_0(x) \;=\; -\rho g x_3 + p_a\,, \tag{3.2}$$

where p_a is a constant external pressure.

We consider small motions of the fluid near to the equilibrium state. We assume that the difference $p(t,x)$ between the full pressure $P(t,x)$ in the process of small motions and the static pressure $P_0(x)$ is an infinitesimal function. We denote the small velocity field in the fluid by $\mathbf{u}(t,x)$ and the vertical displacement of the free surface from the equilibrium by $\zeta(t,\hat{x})$, $\hat{x} = (x_1, x_2) \in \Gamma$. We suppose that the unknown functions $\mathbf{u}(t,x)$, $p(t,x)$ and $\zeta(t,\hat{x})$ are infinitely differentiable with respect to the first variable.

Consider the linearized Navier–Stokes equation which describes the small motions of a viscous fluid, and suitable boundary conditions and initial data. The initial boundary value problem has the form (see, for instance, [23, p. 276]):

$$\frac{\partial\mathbf{u}}{\partial t} \;=\; -\rho^{-1}\nabla p + \nu\Delta\mathbf{u} + \mathbf{f}\,, \quad \operatorname{div}\mathbf{u} \;=\; 0 \quad (\text{in } \Omega)\,, \tag{3.3}$$

$$\mathbf{u} \;=\; \mathbf{0} \quad (\text{on } S)\,, \quad \frac{\partial\zeta}{\partial t} \;=\; u_{\mathbf{n}} \;:=\; \mathbf{u}\cdot\mathbf{n} \quad (\text{on } \Gamma)\,, \quad \int_\Gamma \zeta\, d\Gamma \;=\; 0\,, \tag{3.4}$$

$$\nu\left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right) \;=\; 0 \quad (i = 1,2;\ \text{on } \Gamma)\,, \quad -p + 2\rho\nu\frac{\partial u_3}{\partial x_3} \;=\; -\rho g\zeta \quad (\text{on } \Gamma)\,, \tag{3.5}$$

$$\mathbf{u}(0,x) \;=\; \mathbf{u}^0(x)\,, \quad x \in \Omega\,; \quad \zeta(0,\hat{x}) \;=\; \zeta^0(\hat{x})\,, \quad \hat{x} \in \Gamma\,. \tag{3.6}$$

Here $\mathbf{f} = \mathbf{f}(t, x)$ is a small field of external mass forces which acts together with the gravitational field, $\mathbf{n}$ is the unit vector of the external normal to the boundary $\partial\Omega$. We call the first condition in (3.4) the adhesion condition of the particles of the viscous fluid to the immovable rigid wall S. Respectively, we call the second relation in (3.4) the kinematic condition. It means for small motions of the fluid that a particle of the fluid, which is located on the moving free surface at some moment, remains there also in the future. The last condition in (3.4) follows from the conservation of the fluid volume during the motion. Further, the first condition in (3.5) means that the tangential stresses in the viscous fluid vanish because such stresses are equal to zero in the exterior medium. The last condition in (3.5) follows from the vanishing of the normal stress on the moving free surface of the fluid. Finally, the conditions (3.6) give us the initial velocity field in the domain Ω and the initial displacement of the free surface from the equilibrium Γ.

3.2 Orthogonal subspaces and auxiliary boundary value problems

We shall investigate the problem (3.3) – (3.6) by transforming it to a Cauchy problem for a differential operator equation in some Hilbert space. To this end we introduce the Hilbert spaces and some subspaces which will be used later, and we also consider two auxiliary boundary value problems.

We suppose that the boundary $\partial\Omega$ is a piecewise smooth surface with nonzero inner and outer angles. We assume that the unknown fields $\mathbf{u}(t, x)$ and $\nabla p(t, x)$ are functions of t with values in the Hilbert space $\mathbf{L}_2(\Omega)$ of vector–functions with scalar product

$$(\mathbf{u}, \mathbf{v}) \ := \ \int_\Omega \mathbf{u}(x) \cdot \overline{\mathbf{v}(x)}\, d\Omega\,.$$

We assume that the unknown function $\zeta(t, \hat{x})$ is a function of t with values in the Hilbert space $L_2(\Gamma)$ with scalar product

$$(\zeta, \eta)_0 \ := \ \int_\Gamma \zeta(\hat{x})\overline{\eta(\hat{x})}\, d\Gamma\,.$$

The last condition in (3.4) shows that for each t $\zeta(t, \hat{x})$ belongs to the Hilbert space $H := L_2(\Gamma) \ominus \{1_\Gamma\}$ of functions from $L_2(\Gamma)$ that are orthogonal to the function 1_Γ identically equal to 1 on Γ.

In the sequel, we use the orthogonal decomposition

$$\mathbf{L}_2(\Omega) \ = \ \mathbf{J}_0(\Omega) \oplus \mathbf{G}_{h,S}(\Omega) \oplus \mathbf{G}_{0,\Gamma}(\Omega)\,, \tag{3.7}$$

which is naturally connected with the problem on small oscillations of a viscous fluid in a vessel (see Subsection 2.6, formulae (2.69) – (2.77)).

Suppose that the problem (3.3) – (3.6) has a solution for which all terms in the first equation of (3.3) are continuous functions of t with values in $\mathbf{L}_2(\Omega)$. We conclude from the equations (3.3) and the boundary conditions (3.4) that

$$\mathbf{u}(t, .) \in \mathbf{J}_{0,S}(\Omega) \ = \ \mathbf{G}_{h,S}(\Omega) \oplus \mathbf{J}_0(\Omega), \tag{3.8}$$

$$\nabla p(t, .) \in \mathbf{G}(\Omega) \ = \ \mathbf{G}_{0,\Gamma}(\Omega) \oplus \mathbf{G}_{h,S}(\Omega)\,. \tag{3.9}$$

Further, we introduce the orthoprojections $P_{0,S}$ and $P_{0,\Gamma}$ from $\mathbf{L}_2(\Omega)$ onto the subspaces $\mathbf{J}_{0,S}(\Omega)$ and $\mathbf{G}_{0,\Gamma}(\Omega)$, respectively ($P_{0,S} + P_{0,\Gamma} = I$). Then

$$\begin{aligned} P_{0,S}\mathbf{u}(t, x) \ &= \ \mathbf{u}(t, x)\,, \quad P_{0,S}\nabla p(t, x) \ = \ \nabla\tilde{p}(t, x) \in \mathbf{G}_{h,S}(\Omega)\,, \\ P_{0,\Gamma}\nabla p(t, x) \ &= \ \nabla\varphi(t, x) \in \mathbf{G}_{0,\Gamma}(\Omega)\,. \end{aligned} \tag{3.10}$$

Applying the orthoprojections $P_{0,\Gamma}$ and $P_{0,S}$ to both sides of the first equation in (3.3), we obtain that

$$\mathbf{0} \ = \ -\rho^{-1}\nabla\varphi + \nu P_{0,\Gamma}\Delta\mathbf{u} + P_{0,\Gamma}\mathbf{f}\,, \tag{3.11}$$

$$\frac{\partial\mathbf{u}}{\partial t} \ = \ -\rho^{-1}\nabla\tilde{p} + \nu P_{0,S}\Delta\mathbf{u} + P_{0,S}\mathbf{f}\,. \tag{3.12}$$

The relation (3.11) shows that the field $\nabla\varphi$ is calculated immediately if we know the velocity field $\mathbf{u}$. Thus in the sequel, we shall investigate the equation (3.12), which does not contain $\nabla\varphi$, and also the boundary conditions (3.4), (3.5) (with the change $p \mapsto \tilde{p}$ because $p = \tilde{p} + \varphi$, $\varphi = 0$ (on Γ), see (2.75)) and the initial data (3.6).

Remark 3.1 If the field $\mathbf{u}$ is solenoidal and $u_n = 0$ (on S), then

$$\int_\Gamma u_n \, d\Gamma \;=\; \int_\Gamma u_3 \, d\Gamma \;=\; 0\,.$$

The same property holds also for any surface Γ_α described by the equation $x_3 = -\alpha$, where α is a small positive number. Hence for the classical solution $\mathbf{u}$ of the problem (3.3) – (3.6) we have

$$\int_\Gamma \frac{\partial u_3}{\partial x_3} \, d\Gamma \;=\; 0\,.$$

Taking into account the last condition in (3.4), we obtain from the last condition in (3.5) and the above–mentioned relation that $\int_\Gamma p \, d\Gamma = 0$. Therefore $\nabla p \in \mathbf{G}(\Omega)$ (see (3.9)) for the problem (3.3) – (3.6).

Now consider auxiliary boundary value problems and corresponding operators. As in Subsection 2.3 we introduce the Hilbert space $H^1_\Gamma(\Omega)$ (see (2.25), (2.26)) and the space $H^{\frac{1}{2}}_\Gamma := H^{\frac{1}{2}}(\Gamma) \cap H$, $H := L_2(\Gamma) \ominus \{1_\Gamma\}$ with the norm (2.39).

Auxiliary problem 1° (Zaremba problem for the Laplace operator, see Lemma 2.4). *Solve the following boundary value problem in the unknown function $p_1 \in H^1_{h,S}(\Omega) \subset H^1_\Gamma(\Omega)$:*

$$\Delta p_1 \;=\; 0 \quad (in\ \Omega)\,, \qquad \frac{\partial p_1}{\partial \mathbf{n}} \;=\; 0 \quad (on\ S)\,, \quad p_1 \;=\; \psi \quad (on\ \Gamma) \tag{3.13}$$

for $\psi \in H$, i.e., $\psi \in \mathbf{L}_2(\Gamma)$ such that $\int_\Gamma \psi \, d\Gamma = 0$.

Lemma 3.2 *For any $\psi \in H^{\frac{1}{2}}_\Gamma$ the auxiliary problem* 1° *has a unique solution $p_1 \in H^1_{h,S}(\Omega) \subset H^1_\Gamma(\Omega)$ and $G\psi := \nabla p_1 \in \mathbf{G}_{h,S}(\Omega)$. The operator $G : H^{\frac{1}{2}}_\Gamma \to \mathbf{G}_{h,S}(\Omega)$ is an isometry. The same operator $G : H = L_2(\Gamma) \ominus \{1_\Gamma\} \to \mathbf{G}_{h,S}(\Omega) \subset \mathbf{J}_{0,S}(\Omega)$ is an unbounded closed operator with dense domain $\mathcal{D}(G) := H^{\frac{1}{2}}_\Gamma$.*

Proof. The unique existence of the solution $p_1 \in H^1_\Gamma(\Omega)$ was proved in Lemma 2.4. Further, the formulae (2.67) and (2.41) together with the considerations of the Subsections 2.4 and 2.6 yield that $\nabla p_1 \in \mathbf{G}^1_{h,S}(\Omega)$ and

$$\|\nabla p_1\| \;=\; \|G\psi\| \;=\; \|p_1\|_{H^1_\Gamma(\Omega)} \;=\; \|\psi\|_{H^{\frac{1}{2}}_\Gamma}\,, \tag{3.14}$$

i.e., $G : H^{\frac{1}{2}}_\Gamma \to \mathbf{G}_{h,S}(\Omega)$ is an isometry.

Consider now the operator G as an operator $G : \mathcal{D}(G) = H^{\frac{1}{2}}_\Gamma \subset H \to \mathbf{G}_{h,S}(\Omega)$ and check that G is a closed operator. G can be represented in the form $G = \nabla \circ G_0$ where $\nabla : H^1_{h,S}(\Omega) \to \mathbf{G}_{h,S}(\Omega)$ is a isometry (see (3.14)) and

$$G_0 \;:\; \mathcal{D}(G_0) \;=\; H^{\frac{1}{2}}_\Gamma \subset H \to H^1_{h,S}(\Omega) \subset H^1_\Gamma(\Omega)\,, \quad G_0\psi \;:=\; p_1 \in H^1_{h,S}(\Omega) \subset H^1_\Gamma(\Omega)\,.$$

Evidently, it is sufficient to prove the closure of the operator G_0.

Let $\{\psi_k\}_{k=1}^\infty \subset \mathcal{D}(G_0)$ such that $\psi_k \to \psi$ in H and $p_k := G_0\psi_k \to w \in H^1_\Gamma(\Omega)$. We must prove that $\psi \in \mathcal{D}(G_0)$ and $G_0\psi = w$. We know that $p_k = G_0\psi_k \in H^1_{h,S}(\Omega) \subset H^1_\Gamma(\Omega)$. Since $H^1_{h,S}(\Omega)$ is a subspace of the space $H^1_\Gamma(\Omega)$ (see (2.37)) we obtain that $w = \lim_{k\to\infty} p_k \in H^1_{h,S}(\Omega)$. The isometry property for elements from $H^1_{h,S}(\Omega)$ and $H^{\frac{1}{2}}_\Gamma$ (see Lemma 2.4) yields that

$$w \;=\; G_0\widetilde{\psi}\,, \quad \widetilde{\psi} \in H^{\frac{1}{2}}_\Gamma\,, \qquad \|w\|_{H^1_\Gamma(\Omega)} \;=\; \big\|G_0\widetilde{\psi}\big\|_{H^1_\Gamma(\Omega)} \;=\; \big\|\widetilde{\psi}\big\|_{H^{\frac{1}{2}}_\Gamma}\,.$$

Since the operator $G_0^{-1} : H^1_{h,S}(\Omega) \to H_\Gamma^{\frac{1}{2}}$ is also an isometry we conclude that

$$\psi_k \;=\; G_0^{-1} p_k \;\longrightarrow\; G_0^{-1} w \;=\; \widetilde{\psi} \in H_\Gamma^{\frac{1}{2}} \qquad \Big(\text{lim in } H_\Gamma^{\frac{1}{2}}\Big)$$

and therefore $\|\psi_k - \widetilde{\psi}\|_{H_\Gamma^{\frac{1}{2}}} \to 0$ $(k \to \infty)$. Remember that the space $H_\Gamma^{\frac{1}{2}}$ is densely and compactly embedded into H. Hence, $\|\phi\|_H \le b\,\|\phi\|_{H_\Gamma^{\frac{1}{2}}}$ for all $\phi \in H_\Gamma^{\frac{1}{2}}$. It follows that

$$\|\psi_k - \widetilde{\psi}\|_H \;\le\; b\,\|\psi_k - \widetilde{\psi}\|_{H_\Gamma^{\frac{1}{2}}} \;\longrightarrow\; 0 \quad (k \to \infty)\,.$$

Consequently, $\psi_k \to \widetilde{\psi}$ and $\psi_k \to \psi$ in H. This means that $\psi = \widetilde{\psi} = G_0^{-1} w \in H_\Gamma^{\frac{1}{2}} = \mathcal{D}(G_0)$ and $G_0\psi = w$. □

Now we introduce the subspace

$$\mathbf{J}^1_{0,S}(\Omega) \;:=\; \big\{\mathbf{u} \in \mathbf{H}^1(\Omega) : \operatorname{div}\mathbf{u} = 0 \text{ (in } \Omega),\; \mathbf{u} = \mathbf{0} \text{ (on } S)\big\} \tag{3.15}$$

of the vector field space $\mathbf{H}^1(\Omega)$ and associate the quadratic form

$$E(\mathbf{u},\mathbf{u}) \;:=\; \frac{1}{2}\int_\Omega \sum_{i,j=1}^{3} \left|\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right|^2 d\Omega\,,$$

and the corresponding bilinear one $E(\mathbf{u},\mathbf{v})$ with it. If $\mathbf{u}$ denotes the velocity field in the viscous fluid, the form $\rho\nu E(\mathbf{u},\mathbf{u})$ is equal to the dissipation velocity of the energy in the domain Ω. We mention that on the subspace $\mathbf{J}^1_{0,S}(\Omega)$ the norm in the form $[E(\mathbf{u},\mathbf{u})]^{\frac{1}{2}}$ is equivalent to the standard norm of the space $\mathbf{H}^1(\Omega)$ (see [23, Section 2.2]).

Auxiliary problem 2° (The first auxiliary S. Krein problem). *Solve the following boundary value problem in the unknown functions* $\mathbf{u} \in \mathbf{J}^1_{0,S}(\Omega)$ *and* $\nabla p_2 \in \mathbf{G}_{h,S}(\Omega)$*:*

$$\begin{aligned}
&-\nu P_{0,S}\Delta\mathbf{u} + \rho^{-1}\nabla p_2 \;=\; \mathbf{f}_1\,, \quad \operatorname{div}\mathbf{u} \;=\; 0 \quad (\textit{in}\ \Omega)\,,\\
&\mathbf{u} \;=\; \mathbf{0} \quad (\textit{on}\ S)\,, \quad \nu\left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right) \;=\; 0 \quad (i = 1,2; \textit{on}\ \Gamma)\,,\\
&-p_2 + 2\rho\nu\,\frac{\partial u_3}{\partial x_3} \;=\; 0 \quad (\textit{on}\ \Gamma)\,.
\end{aligned} \tag{3.16}$$

Before we investigate the Problem 2° we give some preliminary explanations. Let $\mathbf{u} \in \mathbf{J}^1_{0,S}(\Omega) \cap \mathbf{H}^2(\Omega)$. By the embedding theorem we conclude that $(\partial u_3/\partial x_3)\big|_\Gamma \in H_\Gamma^{\frac{1}{2}}$ since $\mathbf{u} \in \mathbf{H}^2(\Omega)$. For an unknown field $\nabla p_2 \in \mathbf{G}_{h,S}(\Omega)$ the definition of the space $\mathbf{G}_{h,S}(\Omega)$ yields that the functions $\mathbf{u}$ and ∇p_2 are a solution of (3.16) iff p_2 is a solution of the following Zaremba problem:

$$\Delta p_2 \;=\; 0 \quad (\text{in } \Omega)\,, \qquad \frac{\partial p_2}{\partial \mathbf{n}} \;=\; 0 \quad (\text{on } S)\,, \qquad p_2 \;=\; 2\rho\nu\,\frac{\partial u_3}{\partial x_3} \quad (\text{on } \Gamma)\,.$$

By Lemma 3.2 we obtain that

$$\rho^{-1}\nabla p_2 \;=\; 2\nu G\,\frac{\partial u_3}{\partial x_3}\bigg|_\Gamma \;=:\; \nu\,T_1\mathbf{u}\,, \tag{3.17}$$

and we can write the first equation in formula (3.16) in the form

$$\nu\hat{A}\mathbf{u} \;:=\; -\nu\big[P_{0,S}\Delta\mathbf{u} - T_1\mathbf{u}\big] \;=\; \mathbf{f}_1\,. \tag{3.18}$$

Here the operator $\hat{A}$ is defined on the set

$$\mathcal{D}(\hat{A}) \;:=\; \left\{ \mathbf{u} \in \mathbf{J}^1_{0,S}(\Omega) \cap \mathbf{H}^2(\Omega) \,:\, \frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i} = 0 \;(i = 1, 2;\ \text{on } \Gamma) \right\}, \tag{3.19}$$

which is dense in $\mathbf{J}^1_{0,S}(\Omega)$.

Let $\mathbf{u} \in \mathcal{D}(\hat{A})$, $\mathbf{v} \in \mathbf{J}^1_{0,S}(\Omega)$. Then the following Green's formula is valid (see [23, p. 115]):

$$\int_\Omega \left[-\nu P_{0,S} \Delta \mathbf{u} + \rho^{-1} \nabla p_2 \right] \cdot \overline{\mathbf{v}} \, d\Omega \;=\; \nu E(\mathbf{u}, \mathbf{v}) - \int_\Gamma \left(-\rho^{-1} p_2 + 2\nu \frac{\partial u_3}{\partial x_3} \right) \overline{v_3} \, d\Gamma \,. \tag{3.20}$$

This formula yields for the Problem 2°:

Definition 3.3 A function $\mathbf{u} \in \mathbf{J}^1_{0,S}(\Omega)$ is said to be *a generalized solution* of the Problem 2° if for all $\mathbf{v} \in \mathbf{J}^1_{0,S}(\Omega)$ the identity

$$\nu E(\mathbf{u}, \mathbf{v}) \;=\; (\mathbf{f}_1, \mathbf{v}) \tag{3.21}$$

holds.

Lemma 3.4 *If* $\mathbf{f}_1 \in \mathbf{J}_{0,S}(\Omega)$, *then Problem* 2° *has a unique generalized solution* $\mathbf{u} = \nu^{-1} A^{-1} \mathbf{f}_1$ *where* A *is the operator of the boundary value Problem* 2°. *The operator* A *is a selfadjoint and positive definite* (*strictly positive*) *Friedrichs extension of the operator* $\hat{A}$ *given by* (3.18), (3.19) *in the space* $\mathbf{J}_{0,S}(\Omega)$ *and has the following properties:*

1) $\mathcal{D}(A) \subset \mathcal{D}(A^{\frac{1}{2}}) = \mathbf{J}^1_{0,S}(\Omega) \subset \mathbf{J}_{0,S}(\Omega)$, $\overline{\mathcal{D}(A)} = \mathbf{J}_{0,S}(\Omega)$.

2) *For all* $\mathbf{u} \in \mathcal{D}(A)$, $\mathbf{v} \in \mathbf{J}^1_{0,S}(\Omega)$ *the identity*

$$(A\mathbf{u}, \mathbf{v}) \;=\; E(\mathbf{u}, \mathbf{v})$$

holds. If $\mathbf{u}$, $\mathbf{v} \in \mathbf{J}^1_{0,S}(\Omega)$, *we have*

$$E(\mathbf{u}, \mathbf{v}) \;=\; \left(A^{\frac{1}{2}} \mathbf{u}, A^{\frac{1}{2}} \mathbf{v} \right).$$

3) *The inverse operator* A^{-1} *is compact and positive, and acts in the space* $\mathbf{J}_{0,S}(\Omega)$.

4) *The operator* A *has a discrete spectrum* $\{\lambda_n(A)\}_{n=1}^\infty$ *with accumulation point* $+\infty$ *and with asymptotic behavior*

$$\lambda_n(A) \;=\; \left(\frac{\operatorname{mes} \Omega}{3\pi^2} \right)^{-2/3} n^{2/3} [1 + o(1)] \quad (n \to \infty)\,. \tag{3.22}$$

Proof. See [23, p. 277 – 279]. The asymptotic formula (3.22) was obtained by G. Metivier [36]. □

3.3 Transition to a system of differential operator equations

Return to the initial boundary value problem (3.3) – (3.6) and suppose that it has a classical solution $\{\mathbf{u}(t,x),\, p(t,x),\, \zeta(t,\hat{x})\}$. For the solution the relations (3.10) – (3.12), the boundary conditions (3.4), (3.5) (with the change $p \mapsto \tilde{p}$) and the initial data (3.6) are fulfilled. We also assume that $\mathbf{u}(t,x)$ is an element of the subspace $\mathbf{J}_{0,S}(\Omega)$ for all t, $\nabla \tilde{p}(t,x)$ is an element of the subspace $\mathbf{G}_{h,S}(\Omega)$, and $\zeta(t,\hat{x})$ is an element of the subspace $H = L_2(\Gamma) \ominus \{1_\Gamma\}$.

Represent $\nabla \tilde{p} \in \mathbf{G}_{h,S}(\Omega)$ in the form

$$\nabla \tilde{p} \;=\; \nabla p_1 + \nabla p_2\,, \quad \nabla p_i \in \mathbf{G}_{h,S}(\Omega) \quad (i = 1,\, 2)\,, \tag{3.23}$$

and suppose that p_1 is a solution of the auxiliary Problem 1° for $\psi = \rho g \zeta$, and p_2 is the unknown function from auxiliary Problem 2° (see (3.16)). Then by Lemma 3.2 and definition of the operator G we have

$$\nabla p_1 \;=\; \rho g G \zeta\,. \tag{3.24}$$

Taking into account the formulae (3.23), (3.24), we can rewrite the equation (3.12) in the form

$$\nu\hat{A}\mathbf{u} \;=\; \mathbf{f}_1 \;:=\; -\frac{d\mathbf{u}}{dt} + P_{0,S}\mathbf{f} - gG\zeta\,. \tag{3.25}$$

Here we mention that, by the last boundary condition (3.5), the function $\zeta(t,\hat{x})$ from the classical solution of problem (3.3) – (3.6) is continuously differentiable in its arguments. In particular, the function $\zeta(t,\hat{x})$ belongs to the space $H_\Gamma^{\frac{1}{2}} = H^{\frac{1}{2}}(\Gamma) \cap H$ and $\partial\zeta/\partial t \in H$. Therefore the element $\mathbf{f}_1$ from the right side of Equation (3.25) belongs to the space $\mathbf{J}_{0,S}(\Omega)$, and, by Lemma 3.4, the function $\mathbf{u} = \mathbf{u}(t)$ has the form

$$\mathbf{u} \;=\; \nu^{-1}\,A^{-1}\left(-\frac{d\mathbf{u}}{dt} - gG\zeta + P_{0,S}\mathbf{f}\right). \tag{3.26}$$

As a final result we formulate the following theorem.

Theorem 3.5 *The classical solution of the initial boundary value problem* (3.3) – (3.6) *is the solution of the Cauchy problem for the system of operator differential equations*

$$\begin{aligned} &\frac{d\mathbf{u}}{dt} + \nu A\mathbf{u} + gG\zeta \;=\; P_{0,S}\mathbf{f} \;=:\; \mathbf{f}_{0,S}\,, \\ &\frac{d\zeta}{dt} - \gamma_n\mathbf{u} \;=\; 0\,, \quad \mathbf{u}(0) \;=\; \mathbf{u}^0\,, \quad \zeta(0) \;=\; \zeta^0\,, \end{aligned} \tag{3.27}$$

and for the solution relation (3.11) *is fulfilled. Here A is the operator of Problem* $2°$*, G is the operator connected with Problem* $1°$*, and γ_n is the trace operator of the normal velocity component:*

$$\gamma_n\mathbf{u} \;:=\; u_{\mathbf{n}} \;=\; \mathbf{u}\cdot\mathbf{n}\big|_\Gamma \quad (\mathbf{u} \in \mathbf{J}_{0,S}(\Omega))\,. \tag{3.28}$$

Consider properties of the operator γ_n defined on $\mathbf{J}_{0,S}^1(\Omega)$:

Lemma 3.6 *The operator γ_n with domain*

$$\mathcal{D}(\gamma_n) \;:=\; \mathbf{J}_{0,S}^1(\Omega) \;\subset\; \mathbf{J}_{0,S}(\Omega) \tag{3.29}$$

has the following properties:

$$\mathbf{J}_{0,S}^1(\Omega) \;\subset\; \mathcal{D}(G^*)\,, \quad \gamma_n \;=\; G^*|_{\mathbf{J}_{0,S}^1(\Omega)}\,. \tag{3.30}$$

Proof. Let $\mathbf{v} \in \mathbf{J}_{0,S}^1(\Omega)$ and $\psi \in H_\Gamma^{\frac{1}{2}}$. Lemma 3.2 yields that $\nabla p_1 := G\psi \in \mathbf{G}_{h,S}(\Omega)$ and

$$\begin{aligned} (\nabla p_1, \mathbf{v}) \;&=\; \int_\Omega \nabla p_1 \cdot \overline{\mathbf{v}}\, d\Omega \;=\; \int_\Omega \operatorname{div}(p_1\overline{\mathbf{v}})\, d\Omega \;=\; \int_{\partial\Omega} p_1\, \overline{v_{\mathbf{n}}}\, dS \\ &=\; \int_\Gamma p_1\, \overline{v_{\mathbf{n}}}\, d\Gamma \;=\; (p_1, \gamma_n\mathbf{v})_0 \;=\; (\psi, \gamma_n\mathbf{v})_0\,. \end{aligned}$$

From this it follows that

$$(G\psi, \mathbf{v}) \;=\; (\psi, \gamma_n\mathbf{v})_0\,, \quad \psi \in H_\Gamma^{\frac{1}{2}}\,, \quad \mathbf{v} \in \mathbf{J}_{0,S}^1(\Omega)\,, \tag{3.31}$$

and, therefore, the relations (3.30) hold. □

Remark 3.7 The operator γ_n is equal to the zero operator on the set $\mathbf{J}_0(\Omega) \cap \mathbf{J}_{0,S}^1(\Omega)$, i.e.,

$$\operatorname{Ker}\gamma_n \;=\; \mathbf{J}_0(\Omega) \cap \mathbf{J}_{0,S}^1(\Omega)\,.$$

Remark 3.8 It follows from the proof of Lemma 3.6 that the operator γ_n can be extented to the operator $\widetilde{\gamma}_n$ with domain $\mathcal{D}(\widetilde{\gamma}_n) = \mathbf{J}_{0,S}(\Omega)$. Indeed, relation (3.31) holds if on the right–hand side $\langle\psi, \widetilde{\gamma}_n\mathbf{v}\rangle$ stands for $\psi \in H_\Gamma^{\frac{1}{2}}$ and $\widetilde{\gamma}_n\mathbf{v} \in H_\Gamma^{-\frac{1}{2}} = \left(H_\Gamma^{\frac{1}{2}}\right)^*$, i.e., $\langle\psi, \widetilde{\gamma}_n\mathbf{v}\rangle$ is the value of the functional $\widetilde{\gamma}_n\mathbf{v}$ on the element $\psi \in H_\Gamma^{\frac{1}{2}}$ in the scalar product of the space $H = L_2(\Gamma) \ominus \{1_\Gamma\}$. This property holds for any $\mathbf{v} \in \mathbf{J}_{0,S}(\Omega)$. Then from Remark 3.7 it follows that $\operatorname{Ker}\widetilde{\gamma}_n = \mathbf{J}_0(\Omega)$.

Lemma 3.6 yields:

Corollary 3.9 *For the operators* A *and* γ_n *the following inclusions hold:*

$$\mathcal{D}(A) \subset \mathcal{D}\big(A^{\frac{1}{2}}\big) = \mathcal{D}(\gamma_n) = \mathbf{J}^1_{0,S}(\Omega) \subset \mathcal{D}(G^*)\,. \tag{3.32}$$

Thus the initial boundary value problem (3.3) – (3.6) is decomposed into the trivial relation (3.11) and the Cauchy problem for a system of differential operator equations which can be rewritten in the form

$$\frac{d}{dt}\begin{pmatrix}\mathbf{u}\\ \zeta\end{pmatrix} + \begin{pmatrix}\nu A & gG\\ -G^* & 0\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \zeta\end{pmatrix} = \begin{pmatrix}\mathbf{f}_{0,S}\\ 0\end{pmatrix}, \quad \begin{pmatrix}\mathbf{u}(0)\\ \zeta(0)\end{pmatrix} = \begin{pmatrix}\mathbf{u}^0\\ \zeta^0\end{pmatrix}. \tag{3.33}$$

Here $\mathbf{u} = \mathbf{u}(t)$ and $\zeta = \zeta(t)$ are unknown functions (the velocity field of the fluid and the displacement field of the free surface) with values in $\mathbf{J}_{0,S}(\Omega)$ and $H = L_2(\Gamma) \ominus \{1_\Gamma\}$, respectively.

Definition 3.10 The functions $\mathbf{u}(t,x)$, $\zeta(t,\hat{x})$ and $p(t,x) = \varphi(t,x) + \tilde{p}(t,x)$ are said to be *a strong solution* of the problem (3.3) – (3.6) if the relation (3.11) holds (in the sense of distributions) and the pair $\{\mathbf{u};\zeta\}$ is a strong solution of the Cauchy problem (3.33) in the space $\mathcal{H} := \mathbf{J}_{0,S}(\Omega) \oplus H$. This means that for all $t \geq 0$ the functions $\mathbf{u} = \mathbf{u}(t) \in \mathcal{D}(A)$, $\zeta = \zeta(t) \in \mathcal{D}(G)$ and the functions $d\mathbf{u}/dt$, $d\zeta/dt$, $A\mathbf{u}(t)$, $G\zeta(t)$ are continuous in t, and the equation and the initial data (3.33) hold.

3.4 Investigation of properties of the matrix operator of the boundary value problem

For simplicity we consider problem (3.33) under the assumptions $\nu = 1$, $g = 1$. (If ν and g are arbitrary positive numbers, we can use the transformations $g^{\frac{1}{2}}\zeta \mapsto \eta$, $\nu A \mapsto A$, $g^{\frac{1}{2}}G \mapsto G$.)

We deal with the Cauchy problem

$$\frac{d}{dt}\begin{pmatrix}\mathbf{u}\\ \zeta\end{pmatrix} + \begin{pmatrix}A & G\\ -G^* & 0\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \zeta\end{pmatrix} = \begin{pmatrix}\mathbf{f}_{0,S}\\ 0\end{pmatrix}, \quad \begin{pmatrix}\mathbf{u}(0)\\ \zeta(0)\end{pmatrix} = \begin{pmatrix}\mathbf{u}^0\\ \zeta^0\end{pmatrix}, \tag{3.34}$$

where the operator matrix

$$\mathcal{A}_0 := \begin{pmatrix}A & G\\ -G^* & 0\end{pmatrix}, \tag{3.35}$$

has the domain

$$\mathcal{D}(\mathcal{A}_0) = \mathcal{D}(A) \oplus \mathcal{D}(G) \tag{3.36}$$

which is dense in $\mathcal{H} = \mathbf{J}_{0,S}(\Omega) \oplus H$. By the inclusions (3.32) the operator $\mathcal{A}_0$ is well–defined for any element from $\mathcal{D}(\mathcal{A}_0)$.

Lemma 3.11 *The operator* $\mathcal{A}_0$ *with domain* (3.36) *is an accretive operator, i.e., for all* $(\mathbf{u};\zeta) \in \mathcal{D}(\mathcal{A}_0)$ *the inequality*

$$\operatorname{Re}\left(\mathcal{A}_0\begin{pmatrix}\mathbf{u}\\ \zeta\end{pmatrix}, \begin{pmatrix}\mathbf{u}\\ \zeta\end{pmatrix}\right)_{\mathcal{H}} = (A\mathbf{u},\mathbf{u}) \geq 0 \tag{3.37}$$

holds.

Proof. The statement of the lemma follows by direct calculations. □

Remark 3.12 It follows from the sequel that $\mathcal{A}_0$ is not a maximal accretive operator. In particular, this fact does not allow us to use known results from the theory of contractive semigroups and corresponding theorems on solvability of differential equations in Hilbert spaces.

For (3.34) we define new unknown functions $\mathbf{v}(t)$ and $\eta(t)$ by the transformation

$$\mathbf{u}(t) = e^{at}\mathbf{v}(t), \quad \zeta(t) = e^{at}\eta(t), \quad a > 0. \tag{3.38}$$

Substituting (3.38) in (3.34), we obtain the Cauchy problem

$$\frac{dy}{dt} + \mathcal{A}_a y = f, \quad y(0) = y^0, \tag{3.39}$$

$$y(t) = \begin{pmatrix} \mathbf{v}(t) \\ \eta(t) \end{pmatrix}, \quad f(t) := \begin{pmatrix} \mathbf{f}_{0,S}(t) \\ 0 \end{pmatrix} e^{-at}, \tag{3.40}$$

$$\begin{gathered} y^0 = \begin{pmatrix} \mathbf{v}(0) \\ \eta(0) \end{pmatrix} = \begin{pmatrix} \mathbf{u}^0 \\ \zeta^0 \end{pmatrix}, \\ \mathcal{A}_a := \mathcal{A}_0 + a\mathcal{I} = \begin{pmatrix} A_a & G \\ -G^* & aI \end{pmatrix}, \quad A_a := A + aI, \end{gathered} \tag{3.41}$$

where $\mathcal{I}$ is the identity operator in $\mathcal{H}$.

By (3.37), the operator $\mathcal{A}_a$ is uniformly accretive:

$$\operatorname{Re}(\mathcal{A}_a y, y)_{\mathcal{H}} \geq a\,\|y\|^2_{\mathcal{H}}. \tag{3.42}$$

Hence there exists the inverse operator $\mathcal{A}_a^{-1}$ (on the range of $\mathcal{A}_a$) with the norm

$$\|\mathcal{A}_a^{-1}\| \leq a^{-1}.$$

Further investigation of the problem (3.39) is based on the extension of the operator $\mathcal{A}_a$ to a maximal uniformly accretive operator. Introduce the operators

$$Q := G^* A_a^{-\frac{1}{2}}, \quad Q^+ := A_a^{-\frac{1}{2}} G, \quad \mathcal{D}(Q^+) = \mathcal{D}(G). \tag{3.43}$$

Lemma 3.13 *The following relations hold:*

$$Q^+ = Q^*\big|_{\mathcal{D}(G)}, \quad \overline{Q^+} = Q^* \in \mathfrak{S}_\infty. \tag{3.44}$$

(*We denote by* $\mathfrak{S}_\infty$ *the set of compact operators acting in the Hilbert space.*)

P r o o f. The statements of Lemma 3.13 follow directly from the definition of the adjoint operator and from the representation

$$Q = G^* A_a^{-\frac{1}{2}} = \big(G^* A^{-\frac{1}{2}}\big)\big(A^{\frac{1}{2}} A_a^{-\frac{1}{2}}\big) = \big(\gamma_n A^{-\frac{1}{2}}\big)\big(A^{\frac{1}{2}} A_a^{-\frac{1}{2}}\big) = \widetilde{Q}\big(A^{\frac{1}{2}} A_a^{-\frac{1}{2}}\big), \tag{3.45}$$

where $\widetilde{Q} := \gamma_n A^{-\frac{1}{2}}$ is compact (see [23, p. 282]) and $A^{\frac{1}{2}} A_a^{-\frac{1}{2}}$ bounded. □

Remark 3.14 For the operator $B := \widetilde{Q}^* \widetilde{Q} \in \mathfrak{S}_\infty$ the asymptotic formula

$$\lambda_n(B) = \left(\frac{\operatorname{mes}\Gamma}{16\pi}\right)^{\frac{1}{2}} n^{-\frac{1}{2}}[1 + o(1)] \quad (n \to \infty) \tag{3.46}$$

holds (see [23, p. 283]).

Theorem 3.15 *The operator* $\mathcal{A}_a$ *from* (3.41) *has the following factorizations:*
a) *in Schur–Frobenius form*

$$\mathcal{A}_a = \begin{pmatrix} I & 0 \\ -QA_a^{-\frac{1}{2}} & I \end{pmatrix} \begin{pmatrix} A_a & 0 \\ 0 & aI + QQ^+ \end{pmatrix} \begin{pmatrix} I & A_a^{-\frac{1}{2}} Q^+ \\ 0 & I \end{pmatrix}, \tag{3.47}$$

b) *with symmetrical outer factors*

$$\mathcal{A}_a \; = \; \begin{pmatrix} A_a^{\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & Q^{+} \\ -Q & aI \end{pmatrix} \begin{pmatrix} A_a^{\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} . \tag{3.48}$$

The closure $\mathcal{A}$ of the operator $\mathcal{A}_a$ has the following factorizations:
a) *in Schur–Frobenius form*

$$\mathcal{A} \; = \; \begin{pmatrix} I & 0 \\ -QA_a^{-\frac{1}{2}} & I \end{pmatrix} \begin{pmatrix} A_a & 0 \\ 0 & aI + QQ^{*} \end{pmatrix} \begin{pmatrix} I & A_a^{-\frac{1}{2}}Q^{*} \\ 0 & I \end{pmatrix} , \tag{3.49}$$

b) *with symmetrical outer factors*

$$\mathcal{A} \; = \; \begin{pmatrix} A_a^{\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & Q^{*} \\ -Q & aI \end{pmatrix} \begin{pmatrix} A_a^{\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} . \tag{3.50}$$

The operator $\mathcal{A}$ is defined on the domain

$$\mathcal{D}(\mathcal{A}) \; = \; \left\{ y = \begin{pmatrix} \mathbf{v} \\ \eta \end{pmatrix} \in \mathcal{H} \colon \; \mathbf{v} \in \mathcal{D}\big(A_a^{\frac{1}{2}}\big) , \; \mathbf{v} + A_a^{-\frac{1}{2}} Q^{*} \eta \in \mathcal{D}(A) \right\} \tag{3.51}$$

by the formula

$$\mathcal{A}y \; = \; \begin{pmatrix} A_a(\mathbf{v} + A_a^{-\frac{1}{2}}Q^{*}\eta) \\ -QA_a^{\frac{1}{2}}\mathbf{v} + a\eta \end{pmatrix} , \quad y \in \mathcal{D}(\mathcal{A}) , \tag{3.52}$$

and is a maximal uniformly accretive operator:

$$\operatorname{Re}\,(\mathcal{A}y, y)_{\mathcal{H}} \; \geq \; a\, \|y\|_{\mathcal{H}}^2 , \quad y \in \mathcal{D}(\mathcal{A}) . \tag{3.53}$$

P r o o f. Formulae (3.47) and (3.48) can be verified directly for elements from $\mathcal{D}(\mathcal{A}_a)$. Further, the second and the third factors in the right–hand side of (3.47) can be closed by replacing the operator Q^{+} by Q^{*} (Lemma 3.13). Thus we get the operator $\mathcal{A}$ from (3.49). After this all factors in (3.49) are closed operators having a bounded inverse. Therefore the product (3.49) is a closed operator having a bounded inverse. Hence the operator $\mathcal{A}$ is maximal uniformly accretive and the inequality (3.53) holds (after closing the operator $\mathcal{A}_a$ in (3.42)).

By the same argumentation as above, we can check formulae (3.48) and (3.50). We mention only that the middle factor in (3.50) has the bounded inverse

$$\begin{pmatrix} I & Q^{*} \\ -Q & aI \end{pmatrix}^{-1} \; = \; \begin{pmatrix} \big(I + a^{-1}Q^{*}Q\big)^{-1} & -a^{-1}Q^{*}\big(I + a^{-1}QQ^{*}\big)^{-1} \\ a^{-1}\big(I + a^{-1}QQ^{*}\big)^{-1}Q & a^{-1}\big(I + a^{-1}QQ^{*}\big)^{-1} \end{pmatrix} .$$

□

3.5 Existence theorem for a unique strong solution

Using the properties of the matrix operator $\mathcal{A}$ obtained above, we now prove a theorem on the well–posed solvability of the problem (3.3) – (3.6).

Theorem 3.16 *If the conditions*

$$\mathbf{u}^0 \in \mathcal{D}(A) , \quad \zeta^0 \in \mathcal{D}(G) \; = \; H_{\Gamma}^{\frac{1}{2}} , \quad \mathbf{f}(t) \in C^1[0, T; \mathbf{L}_2(\Omega)] \tag{3.54}$$

are fulfilled for the Cauchy problem (3.39) – (3.41)*, then it has a unique strong solution for $t \in [0, T]$.*

P r o o f. Instead of the problem (3.39) – (3.41) we consider the Cauchy problem

$$\frac{dy}{dt} + \mathcal{A}y \;=\; f\,, \quad y(0) \;=\; y^0 \;=\; \begin{pmatrix} \mathbf{u}^0 \\ \zeta^0 \end{pmatrix}. \tag{3.55}$$

The first two conditions in (3.54) show that

$$y^0 \;=\; \begin{pmatrix} \mathbf{v}(0) \\ \eta(0) \end{pmatrix} \in \mathcal{D}(\mathcal{A}_0) \;=\; \mathcal{D}(\mathcal{A}_a) \subset \mathcal{D}(\mathcal{A})\,. \tag{3.56}$$

It follows from the last condition in (3.54) that $\mathbf{f}_{0,S}(t) := P_{0,S}\mathbf{f}(t) \in C^1[0,T;\mathbf{J}_{0,S}(\Omega)]$, and therefore by (3.40)

$$f(t) \;=\; \begin{pmatrix} \mathbf{f}_{0,S}(t) \\ 0 \end{pmatrix} e^{-at} \in C^1[0,T;\mathcal{H}]\,. \tag{3.57}$$

Since the operator $\mathcal{A}$ is maximal uniformly accretive acting in the Hilbert space $\mathcal{H}$ (Theorem 3.15), the problem (3.55) has a unique strong solution y on the segment $[0,T]$ fulfilling the conditions (3.56), (3.57) (see, for instance, [30, p. 166], and also [31], [14], [26]). In this case the operator $(-\mathcal{A})$ is a generator of a contractive semigroup $U(t) := \exp(-t\mathcal{A})$ and, by (3.53), the estimate

$$\|U(t)\| \;\le\; e^{-at} \tag{3.58}$$

holds. The solution y can be calculated by the formula

$$y(t) \;=\; U(t)y^0 + \int_0^t U(t-s)f(s)\,ds\,. \tag{3.59}$$

For $y(t) =: (\mathbf{v}(t);\, \eta(t))$ the equation (3.55) holds for all $t \in [0,T]$, i.e., the following equations and initial data are valid:

$$\begin{aligned} &\frac{d\mathbf{v}}{dt} + A_a\big(\mathbf{v} + A_a^{-\frac{1}{2}}Q^*\eta\big) \;=\; \mathbf{f}_{0,S}e^{-at}\,, \quad \mathbf{v}(0) \;=\; \mathbf{u}^0\,, \\ &\frac{d\eta}{dt} - QA_a^{\frac{1}{2}}\mathbf{v} + a\eta \;=\; 0\,, \quad \eta(0) \;=\; \zeta^0\,. \end{aligned} \tag{3.60}$$

Here each term is a continuous function in t with values in $\mathbf{J}_{0,S}(\Omega)$ (in the first equation) and in H (in the second one), respectively.

The second equation yields

$$\eta(t) \;=\; \zeta^0 e^{-at} + \int_0^t e^{-a(t-s)}QA_a^{\frac{1}{2}}\mathbf{v}(s)\,ds\,. \tag{3.61}$$

Substituting this function into the first equation (3.60), we obtain

$$\frac{d\mathbf{v}}{dt}(t) + A_a\left[\mathbf{v}(t) + e^{-at}A_a^{-\frac{1}{2}}Q^*\zeta^0 + A_a^{-\frac{1}{2}}Q^*\int_0^t e^{-a(t-s)}QA_a^{\frac{1}{2}}\mathbf{v}(s)\,ds\right] \;=\; \mathbf{f}_{0,S}(t)e^{-at}\,. \tag{3.62}$$

We conclude from the above (see (3.51), (3.52)) that the function

$$\mathbf{v}(t) + e^{-at}A_a^{-\frac{1}{2}}Q^*\zeta^0 + A_a^{-\frac{1}{2}}Q^*\int_0^t e^{-a(t-s)}QA_a^{\frac{1}{2}}\mathbf{v}(s)\,ds \;=:\; \mathbf{v}_a(t) \tag{3.63}$$

belongs to $\mathcal{D}(A_a)$ for all $t \in [0,T]$ and $A_a\mathbf{v}_a(t) \in C[0,T;\mathbf{J}_{0,S}(\Omega)]$. By (3.54), i.e., $\zeta^0 \in \mathcal{D}(G)$, and Lemma 3.13 we obtain $Q^*\zeta^0 = Q^+\zeta^0 = A_a^{-\frac{1}{2}}G\zeta^0$ and therefore, $A_a^{-\frac{1}{2}}Q^*\zeta^0 = A_a^{-1}G\zeta^0 \in \mathcal{D}(A_a)$.

Consider the operator $T := A_a^{-\frac{1}{2}}Q^*QA_a^{\frac{1}{2}}$ $\big(\mathcal{D}(T) = \mathcal{D}\big(A_a^{\frac{1}{2}}\big)\big)$ (see (3.62), (3.43) and Lemma 3.6). Introduce $\mathcal{D}(A_a)$ as a Hilbert space with graph norm

$$\|\mathbf{v}\|_{\mathcal{D}(A_a)} \;:=\; \|A_a\mathbf{v}\|\,. \tag{3.64}$$

Then the restriction $T_a := T\big|_{\mathcal{D}(A_a)}$ is a linear bounded operator acting in $\mathcal{D}(A_a)$. Indeed, if $\mathbf{v} \in \mathcal{D}(A_a) \subset \mathcal{D}\big(A_a^{\frac{1}{2}}\big) = \mathbf{J}^1_{0,S}(\Omega)$, then, by Gagliardo's embedding theorem [12], $\gamma_n \mathbf{v} \in H^{\frac{1}{2}}_\Gamma = \mathcal{D}(G)$, and Lemma 3.13 yields that

$$T_a \mathbf{v} \;:=\; A_a^{-\frac{1}{2}} Q^* \gamma_n \mathbf{v} \;=\; A_a^{-\frac{1}{2}} Q^+ \gamma_n \mathbf{v} \;=\; A_a^{-1} G \gamma_n \mathbf{v} \in \mathcal{D}(A_a)\,.$$

Since $G\gamma_n$ is a bounded operator from $\mathbf{J}^1_{0,S}(\Omega) = \mathcal{D}\big(A^{\frac{1}{2}}\big)$ into $\mathbf{J}_{0,S}(\Omega)$, we conclude that $T_a\colon\ \mathcal{D}(A_a) \to \mathcal{D}(A_a)$ is a linear bounded operator.

The proved fact allows to consider the relation (3.63) as a linear integral Volterra equation of the second kind in the space $\mathcal{D}(A_a)$ (with graph norm (3.64)). Here the known function $\mathbf{v}_a(t) - e^{-at} A_a^{-\frac{1}{2}} Q^* \zeta^0 \in C[0,T;\mathcal{D}(A_a)]$, and the kernel function $T_a e^{-a(t-s)}$ of the integral operator is a continuous function in the variables $t,\ s$ and bounded on $\mathcal{D}(A_a)$. Therefore problem (3.63) has a unique solution $\mathbf{v}(t) \in C[0,T;\mathcal{D}(A_a)]$ and each term in (3.63) is an element from $C[0,T;\mathcal{D}(A_a)]$.

Thus in the equation (3.62) and in the first equation (3.60) we can open the brackets. Since the operator A_a is bounded and acts from $\mathcal{D}(A_a)$ into $\mathbf{J}_{0,S}(\Omega)]$, we obtain from (3.60) that the equation (3.39) holds for the function $y(t) = (\mathbf{v}(t);\, \eta(t))$. In this we take into account that formula (3.61) and the property $\mathbf{v} \in \mathcal{D}(A_a)$ yield that $\eta(t) \in \mathcal{D}(G)$. □

As a corollary of Theorem 3.16 we have the following result.

Theorem 3.17 *If the conditions* (3.54) *hold, then the problem* (3.3) – (3.6) *has a unique strong solution* (*in the sense of Subsection* 3.3, *Definition* 3.10) *for all* $t \in [0,T]$. *If* $\mathbf{f}(t,x) \equiv \mathbf{0}$, *then for the solution we have the estimate*

$$\|\mathbf{u}(t,.)\|^2 + \|\zeta(t,.)\|_0^2 \;\le\; \big\|\mathbf{u}^0\big\|^2 + \big\|\zeta^0\big\|_0^2\,. \tag{3.65}$$

P r o o f. The proof is based on the transformation (3.38). The estimate (3.65) follows from the formula (3.59) for $f(s) = 0$ and the estimate (3.58). □

4 Normal oscillation of a viscous fluid

4.1 The problem on normal oscillations

We consider the homogeneous problem (3.55) and its solutions of the form

$$y(t) \;=\; y\exp(-\lambda t)\,, \quad y \in \mathcal{H}\,, \quad \lambda \in \mathbb{C}\,, \tag{4.1}$$

will be called *normal oscillations*. Here λ is a complex frequency of the oscillations and y an amplitude element. Substituting the function (4.1) into the homogeneous equation (3.55), we obtain the spectral problem

$$\mathcal{A}y \;=\; \lambda y\,, \quad y \in \mathcal{D}(\mathcal{A}) \tag{4.2}$$

for the amplitude elements y. Here $\mathcal{A}$ is the matrix operator defined by formulae (3.49) – (3.52). We shall call $\mathcal{A}$ the operator associated with the initial boundary value problem (3.3) – (3.6).

First we deduce some evident properties of the operator $\mathcal{A}$.

Lemma 4.1 *The operator* $\mathcal{A}$ *has a bounded inverse* $\mathcal{A}^{-1}$ *such that*

$$\big\|\mathcal{A}^{-1}\big\| \;\le\; a^{-1}\,. \tag{4.3}$$

The spectrum $\sigma(\mathcal{A})$ *of the operator* $\mathcal{A}$ *belongs to the domain*

$$\{\lambda \in \mathbb{C}\colon\ \operatorname{Re}\lambda \ge a\}\,, \tag{4.4}$$

and its eigenvalues λ *are located in the domain*

$$\operatorname{Re}\lambda \;>\; a\,. \tag{4.5}$$

Proof. The uniform accretiveness (3.53) yields the boundedness of the operator $\mathcal{A}^{-1}$, the estimate (4.3) and the fact that points λ, $\operatorname{Re}\lambda < a$, belong to the resolvent set $\rho(\mathcal{A})$ of the operator $\mathcal{A}$ (see [18, Chap. V.10]). It remains to be proved that the eigenvalues of the operator $\mathcal{A}$ are not located on the line $\lambda = a + ib$ ($b \in \mathbb{R}$). Indeed, let $\lambda = a + ib$ and $y = (\mathbf{v}; \eta) \in \mathcal{D}(\mathcal{A})$ such that $(\mathcal{A} - \lambda)y = 0$. Then $\mathbf{v} \in \mathcal{D}\big(A_a^{\frac{1}{2}}\big)$ and the definition (3.52) of the operator $\mathcal{A}$ yields the following system of equations:

$$A_a^{\frac{1}{2}}\big(A_a^{\frac{1}{2}}\mathbf{v} + Q^*\eta\big) \;=\; (a+ib)\mathbf{v}\,, \quad -QA_a^{\frac{1}{2}}\mathbf{v} \;=\; ib\eta\,. \tag{4.6}$$

Since $A_a^{\frac{1}{2}}$ is selfadjoint, we conclude that

$$\big\|A_a^{\frac{1}{2}}\mathbf{v}\big\|^2 + \big(Q^*\eta, A_a^{\frac{1}{2}}\mathbf{v}\big) \;=\; (a+ib)\,\|\mathbf{v}\|^2\,,$$

$$-\big(A_a^{\frac{1}{2}}\mathbf{v}, Q^*\eta\big) \;=\; -\big(QA_a^{\frac{1}{2}}\mathbf{v}, \eta\big)_0 \;=\; ib\,\|\eta\|_0^2\,.$$

The first equality yields

$$\big\|A_a^{\frac{1}{2}}\mathbf{v}\big\|^2 \;=\; a\,\|\mathbf{v}\|^2 \;=\; a(\mathbf{v},\mathbf{v})$$

since we know from the second one that $\big(Q^*\eta, A_a^{\frac{1}{2}}\mathbf{v}\big)$ is purely imaginary. Since

$$\big\|A_a^{\frac{1}{2}}\mathbf{v}\big\|^2 \;=\; \big\|A^{\frac{1}{2}}\mathbf{v}\big\|^2 + a\,\|\mathbf{v}\|^2$$

we conclude that $0 = \big\|A^{\frac{1}{2}}\mathbf{v}\big\|^2$ and, therefore, $\mathbf{v} = \mathbf{0}$. If $b \neq 0$, then the second equation in formula (4.6) yields $\eta = 0$ and, hence, $y = (\mathbf{v}; \eta) = 0$.

If $b = 0$, then we obtain from the first equation in (4.6) that

$$0 \;=\; \big(Q^*\eta, A_a^{\frac{1}{2}}\mathbf{w}\big) \;=\; \big(\eta, QA_a^{\frac{1}{2}}\mathbf{w}\big)_0 \;=\; (\eta, \gamma_n\mathbf{w})_0$$

for all $\mathbf{w} \in \mathbf{J}_{0,S}^1(\Omega) = \mathcal{D}\big(A_a^{\frac{1}{2}}\big)$. Since the set $\big\{\gamma_n\mathbf{w} :\ \mathbf{w} \in \mathbf{J}_{0,S}^1(\Omega)\big\}$ coincides with the space $H_\Gamma^{\frac{1}{2}}$ and this set is dense in H, it follows from the last equality that $\eta = 0$ and again $y = 0$. □

4.2 General properties of the solutions of the spectral problem

The further investigation is based on the theory of linear operators that are selfadjoint in a Hilbert space with indefinite metric. Here we recall some results of this theory (see [6], [5]).

The Hilbert space $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ is said to be a *Krein space* or *J–space* if for elements from $\mathcal{K}$ an indefinite scalar product $[\,\cdot\,,\cdot\,]$ is introduced along with the ordinary scalar product $(\,\cdot\,,\cdot)$. The corresponding indefinite metric is defined by the formula $[\,\cdot\,,\cdot\,] := (J\cdot\,,\cdot)$, where $J = P^+ - P^-$, $P^\pm$ are orthoprojectors on $\mathcal{K}^\pm$, i.e., $J = \operatorname{diag}\big(I^+; -I^-\big)$.

Each element $y \in \mathcal{K}$ has a certain sign with respect to the indefinite metric: it is positive ($[y,y] > 0$), negative ($[y,y] < 0$) or neutral ($[y,y] = 0$). In a Krein space the sign of a subspace and the concept of order are introduced in a natural way. So, for instance, the subspace $\mathcal{L}_+$ is said to be *nonnegative* if $[y,y] \geq 0$ for all $y \in \mathcal{L}_+$, and it is said to be *maximal nonnegative* if it is not a principal part of any other nonnegative subspace.

In the sequel we shall use the following well–known fact: the subspace $\mathcal{L}_+$ is maximal nonnegative if there exists a contraction $K_+ :\ \mathcal{K}^+ \to \mathcal{K}^-$ ($\|K_+\| \leq 1$) such that

$$\mathcal{L}_+ \;=\; \{y = y_+ + K_+y_+ :\ y_+ \in \mathcal{K}^+\}\,. \tag{4.7}$$

The contraction K_+ is called the *angular operator* of the subspace $\mathcal{L}_+$.

A positive subspace $\mathcal{L}_+$ is said to be *uniformly positive* if it is a Hilbert space with respect to the scalar product generated by the indefinite metric. This is the case if and only if $\|K_+\| < 1$.

We shall say that a subspace $\mathcal{L}_+$ *belongs to the class* h^+ if it can be represented as a sum of an uniformly positive subspace and a finite dimensional neutral one. In particular, $\mathcal{L}_+ \in h^+$ if its angular operator $K_+ \in \mathfrak{S}_\infty$.

Nonpositive, maximal nonpositive subspaces and its angular operators, and also uniformly negative subspaces, subspaces of the class h^-, are introduced in the same way.

If a maximal nonnegative subspace $\mathcal{L}_+$ and a maximal nonpositive subspace $\mathcal{L}_-$ are $\mathcal{J}$–orthogonal, i.e., $[y_+, y_-] = 0$ for all $y_\pm \in \mathcal{L}_\pm$, then we say that $\mathcal{L}_+$ and $\mathcal{L}_-$ form a *dual pair* $\{\mathcal{L}_+, \mathcal{L}_-\}$. In the sequel, we shall write $\{\mathcal{L}_+, \mathcal{L}_-\} \in h$ if $\mathcal{L}_\pm \in h^\pm$.

Let T be a linear densely defined operator acting in the space $\mathcal{K}$. The concepts of the J–adjoint operator $T^{[*]}$, a J–symmetric operator $\big(T \subset T^{[*]}\big)$ and a J–selfadjoint one $\big(T = T^{[*]}\big)$ are introduced in the ordinary way. It follows from the relation $T^{[*]} = JT^*J$ that, in particular, the spectrum of a J–selfadjoint operator is symmetric with respect to the real axis.

We say that a continuous J–selfadjoint operator T belongs to the class (H) if there exists at least one dual pair $\{\mathcal{L}_+, \mathcal{L}_-\}$ of subspaces that are invariant with respect to T, and each T–invariant dual pair belongs to the class h. Let $T \in (H)$ be a bounded J–selfadjoint operator in a Krein space $\mathcal{K}$. Let

$$\kappa_T \;=\; \min\{\dim(\mathfrak{L}_+ \cap \mathfrak{L}_-) \,:\, \{\mathfrak{L}_+, \mathfrak{L}_-\} \in h,\; T\mathfrak{L}_\pm \subset \mathfrak{L}_\pm\}\;.$$

Assume $\alpha = \overline{\alpha} \in \sigma_p(T)$ and let $\mathcal{L}_\alpha(T)$ be the root subspace corresponding to α. Then (see [6, Chap. 4 § 2, p. 228, Chap. 3 § 5, Chap. 2 § 6]) $\mathcal{L}_\alpha(T)$ admits the following decomposition:

$$\mathcal{L}_\alpha(T) \;=\; \mathcal{L}_\alpha^{(1)}(T)[\dot{+}]\mathcal{L}_\alpha^{(2)}(T)\,,$$

where $\mathcal{L}_\alpha^{(1)}(T)$ is a finite dimensional subspace such that $\operatorname{Ker}\Big((T-\alpha I)\big|_{\mathcal{L}_\alpha^{(1)}(T)}\Big)$ is the isotropic part of $\operatorname{Ker}(T - \alpha I)$, that is,

$$\operatorname{Ker}\Big((T-\alpha I)\big|_{\mathcal{L}_\alpha^{(1)}(T)}\Big) \;=\; \operatorname{Ker}(T-\alpha I) \cap (\operatorname{Ker}(T-\alpha I))^{[\perp]}\,,$$

$\mathcal{L}_\alpha^{(2)}(T) \subset \operatorname{Ker}(T - \alpha I)$ and $\mathcal{L}_\alpha^{(2)}(T)$ is a regular subspace. Moreover, $\mathcal{L}_\alpha^{(2)}(T)$ is a Pontryagin subspace with index $\kappa(\alpha)$.

Theorem 4.2 (See [6].) *Let $T \in (H)$ be a bounded J–selfadjoint operator in a Krein space $\mathcal{K}$. Then the nonreal spectrum of T consists of a finite number of eigenvalues $\lambda_1, \overline{\lambda}_1, \ldots, \lambda_{n(T)}, \overline{\lambda}_{n(T)}$, and there is only a finite set of real eigenvalues $\alpha_1, \ldots, \alpha_{p(T)}$ such that $\mathcal{L}_{\alpha_i}(T)$, $i = \overline{1, p(T)}$, contains at least one nonzero neutral eigenvector. Moreover,*

$$\sum_{i=1}^{n(T)} \dim \mathcal{L}_{\lambda_i}(T) + \frac{1}{2}\sum_{j=1}^{p(T)} \dim \mathcal{L}_{\alpha_j}^{(1)}(T) + \sum_{j=1}^{p(T)} \kappa(\alpha_j) \;\le\; \kappa_T\,.$$

Definition 4.3 A point $\lambda \in \mathbb{R}$ is called a *critical point* of the operator $T \in (H)$ if its kernel $\operatorname{Ker}(T - \lambda I)$ is degenerate, i.e., its isotropic part has nonzero elements. $s^0(T)$ denotes the set of critical points of the operator $T \in (H)$.

$\mathcal{F}_0(T)$ denotes the closure of the linear hull of eigenelements of the operator T corresponding to the finite multiple eigenvalues, and $\mathcal{F}(T)$ denotes the closure of the linear hull of root elements, respectively.

Definition 4.4 A basis $\{y_k\}_{k=1}^\infty$ of the J–space $\mathcal{H}$ is said to be *almost J–orthonormal* if it can be represented as the union of a finite subset of elements from $\mathcal{H}$ and a J–orthonormal subset, and these subsets are J–orthogonal to each other.

Definition 4.5 We say that an operator $T \in \mathfrak{S}_\infty$ *belongs to class* $\mathfrak{S}_p$ if its s–numbers $s_k(T)$, i.e., the eigenvalues of the operator $(T^*T)^{\frac{1}{2}}$, fulfill the following condition

$$\sum_{k=1}^\infty [s_k(T)]^p \;=\; \sum_{k=1}^\infty \big[\lambda_k\big((T^*T)^{\frac{1}{2}}\big)\big]^p \;<\; \infty\,.$$

Definition 4.6 A basis $\{\psi_n\}_{n=1}^\infty \subset \mathcal{H}$ is said to be a *Riesz basis* if $\psi_n = T\varphi_n$, where $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal basis in $\mathcal{H}$ and $T, T^{-1} \in \mathcal{L}(\mathcal{H})$, i.e., they are linear bounded operators acting in $\mathcal{H}$. Riesz basis is said to be *p–basis* (see [38]) if $T = I + T_1, T_1 \in \mathfrak{S}_p$.

Theorem 4.7 (Cf. [6, Theorem 4.2.12].) *Let $T \in (H)$ be a bounded J–selfadjoint operator in a Krein space $\mathcal{K}$. Assume that its spectrum is countable. Then:*
(a) $\dim(\mathcal{K}/\mathcal{F}(T)) \leq \dim(\mathcal{K}/\mathcal{F}_0(T)) < \infty$.
(b) $\dim(\mathcal{F}(T)/\mathcal{F}_0(T)) < \infty$.
(c) *$\mathcal{F}_0(T) = \mathcal{K}$ if and only if $s^0(T) = \emptyset$ and $\mathcal{L}_\lambda(T) = \mathrm{Ker}\,(T - \lambda I)$ for nonreal λ's.*
(d) *$\mathcal{F}(T) = \mathcal{K}$ if and only if the subspaces $\mathcal{L}_\lambda(T)$ are nondegenerate for all $\lambda \in s^0(T)$.*
(e) *If $\mathcal{F}_0(T) = \mathcal{K}$ ($\mathcal{F}(T) = \mathcal{K}$, respectively) then there exists an almost J–orthonormal basis in $\mathcal{K}$, which consists of eigenvectors (root vectors, respectively) of T.*
(f) *If $\mathcal{F}_0(T) = \mathcal{K}$ then there exists a J–orthonormal basis in $\mathcal{K}$, which consists of eigenvectors of T if and only if the spectrum of T is real.*
(g) *The above mentioned bases can be chosen as p–bases iff T has a maximal nonnegative invariant subspace whose angular operator belongs to the class $\mathfrak{S}_p$.*

In the sequel, the space $\mathcal{H} = \mathbf{J}_{0,S}(\Omega) \oplus H$ will be considered as the space $\mathcal{K}$ with $\mathcal{K}^+ := \mathbf{J}_{0,S}(\Omega)$ and $\mathcal{K}^- := H$.

Lemma 4.8 *The operator matrix $\mathcal{A}$ from* (3.49)–(3.52) *and its inverse $\mathcal{A}^{-1}$ are $\mathcal{J}$–selfadjoint operators with*

$$\mathcal{J} := \mathrm{diag}\,(I_\Omega; -I_\Gamma)\,, \tag{4.8}$$

where I_Ω and I_Γ denote the identity operators in $\mathbf{J}_{0,S}(\Omega)$ and $H = L_2(\Gamma) \ominus \{1_\Gamma\}$, respectively.

The spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A}$ is symmetrical with respect to the real axis and it is located in the domain (4.4).

P r o o f. The inverse $\mathcal{A}^{-1}$ has the following representation

$$\begin{aligned} \mathcal{A}^{-1} &= \begin{pmatrix} I & -A_a^{-\frac12}Q^* \\ 0 & I \end{pmatrix}\begin{pmatrix} A_a^{-1} & 0 \\ 0 & a^{-1}T_2 \end{pmatrix}\begin{pmatrix} I & 0 \\ QA_a^{-\frac12} & I \end{pmatrix} \\ &= \begin{pmatrix} A_a^{-\frac12} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & Q^* \\ -Q & aI \end{pmatrix}^{-1}\begin{pmatrix} A_a^{-\frac12} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A_a^{-\frac12} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} T_1 & -a^{-1}Q^*T_2 \\ a^{-1}T_2Q & a^{-1}T_2 \end{pmatrix}\begin{pmatrix} A_a^{-\frac12} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A_a^{-\frac12}T_1A_a^{-\frac12} & -a^{-1}A_a^{-\frac12}Q^*T_2 \\ a^{-1}T_2QA_a^{-\frac12} & a^{-1}T_2 \end{pmatrix} \\ &=: \begin{pmatrix} \mathcal{A}_{11}^{(-1)} & \mathcal{A}_{12}^{(-1)} \\ \mathcal{A}_{21}^{(-1)} & \mathcal{A}_{22}^{(-1)} \end{pmatrix}, \end{aligned} \tag{4.9}$$

with

$$T_1 = \left(I + a^{-1}Q^*Q\right)^{-1}, \quad T_2 = \left(I + a^{-1}QQ^*\right)^{-1}.$$

The $\mathcal{J}$–symmetry of the operators $\mathcal{A}$ and $\mathcal{A}^{-1}$ is clear from the representations (3.50) and (4.9). The operator $\mathcal{A}^{-1}$ is bounded and everywhere defined. Hence it is $\mathcal{J}$–selfadjoint. Therefore its inverse $\left(\mathcal{A}^{-1}\right)^{-1} = \mathcal{A}$ is also $\mathcal{J}$–selfadjoint. From this it follows that $\sigma(\mathcal{A})$ and $\sigma\left(\mathcal{A}^{-1}\right)$ are symmetric with respect to the real axis. The property $\mathrm{Re}\,\lambda \geq a > 0$ for $\lambda \in \sigma(\mathcal{A})$ was proved in Lemma 4.1. □

The problem (4.2) is equivalent to the problem

$$\mathcal{A}^{-1}y = \mu y\,, \quad y \in \mathcal{H}\,, \quad \mu = \lambda^{-1}\,. \tag{4.10}$$

Lemma 4.9 *The spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A}$ is located on the half–axis $\lambda \geq a$ with the exception of at most a finite number of nonreal eigenvalues (counted according to their multiplicities).*

P r o o f. According to the representation (4.9), the operator matrix $\mathcal{A}^{-1}$ consists of compact operators $\mathcal{A}_{11}^{(-1)}$, $\mathcal{A}_{12}^{(-1)}$ and $\mathcal{A}_{21}^{(-1)} = -\left(\mathcal{A}_{12}^{(-1)}\right)^*$ and a bounded positive definite operator $\mathcal{A}_{22}^{(-1)}$. In particular, the property $\mathcal{A}_{12}^{(-1)} \in \mathfrak{S}_\infty$ and H. Langer's Theorem (see, for instance, [6, Chap. 3 § 5]) yield that the $\mathcal{J}$–selfadjoint operator $\mathcal{A}^{-1}$ has a dual invariant pair $\{\mathcal{L}_+, \mathcal{L}_-\}$. Let $K = K_+ : \mathbf{J}_{0,S}(\Omega) \to H$ be the angular operator of the invariant subspace $\mathcal{L}_+$. Then $\|K\| \le 1$ and

$$\mathcal{L}_+ = \left\{ \begin{pmatrix} \mathbf{v} \\ K\mathbf{v} \end{pmatrix} \in \mathcal{H} = \mathbf{J}_{0,S}(\Omega) \oplus H : \ \mathbf{v} \in \mathbf{J}_{0,S}(\Omega) \right\}. \tag{4.11}$$

Since $\mathcal{A}^{-1} y \in \mathcal{L}_+$ for any $y = (\mathbf{v}; K\mathbf{v}) \in \mathcal{L}_+$, we obtain the Riccati equation

$$\mathcal{A}_{22}^{(-1)} K = -\mathcal{A}_{21}^{(-1)} + K\mathcal{A}_{11}^{(-1)} + K\mathcal{A}_{12}^{(-1)} K \tag{4.12}$$

for the operator K in the standard way from the representation (4.9).

Since the operator $\mathcal{A}_{22}^{(-1)}$ has a bounded inverse, we conclude from (4.12) that K is compact, i.e., $K \in \mathfrak{S}_\infty$. Therefore, $\{\mathcal{L}_+, \mathcal{L}_-\} \in h$ and $\mathcal{A} \in (H)$. Thus the assertion of the Lemma follows from Theorem 4.2 and the property $\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda \ge a\}$. □

Corollary 4.10 *If $\|K\| < 1$, then the operator $\mathcal{A}$ is similar to a selfadjoint one and therefore its spectrum $\sigma(\mathcal{A}) \subset [a, +\infty)$.*

In fact, in this case $\mathcal{L}_+$ is a Hilbert space with respect to the J–metric $[\,\cdot\,,\cdot\,]$ and its J–orthogonal complement $\mathcal{L}_-$ is a Hilbert space with respect to the $-J$–metric $-[\cdot\,,\,\cdot]$. Since booth subspaces $\mathcal{L}_\pm$ are $\mathcal{A}$–invariant we have that $\mathcal{A}$ is a selfadjoint operator with respect to the scalar product $(x,y)_1 := [x_+, y_+] - [x_-, y_-]$, where $x = x_+ + x_-$, $y_+ + y_-$, $x_\pm, y_\pm \in \mathcal{L}_\pm$. It remains to observe that the norms $(x,x)^{\frac12}$ and $(x,x)_1^{\frac12}$ are equivalent.

Lemma 4.11 *The essential (limiting) spectrum $\sigma_{\mathrm{ess}}(\mathcal{A})$ of the operator $\mathcal{A}$ coincides with the set $\{a\} \cup \{\infty\}$.*

P r o o f. It is sufficient to show that the spectrum $\sigma_{\mathrm{ess}}\left(\mathcal{A}^{-1}\right)$ coincides with the set $\left\{a^{-1}\right\} \cup \{0\}$. But this directly follows from Weyl's Theorem on compact perturbation. Indeed, since the operators $A_{11}^{(-1)}$, $A_{12}^{(-1)}$, $A_{21}^{(-1)}$, Q and Q^* are compact (see Lemmas 3.4 and 3.13) we conclude from the description (4.9) of the operator $\mathcal{A}^{-1}$ that $\mathcal{A}^{-1}$ is a compact perturbation of the operator $\mathrm{diag}\left(0; a^{-1} I_\Gamma\right)$ with essential spectrum $\{0\} \cup \left\{a^{-1}\right\}$. □

Theorem 4.12 *The problem (4.2) has a countable set of positive eigenvalues $\left\{\lambda_k^+\right\}_{k=1}^\infty$ of finite–multiplicity with the accumulation point $\lambda = +\infty$ and eigenelements $\left\{y_k^+\right\}_{k=1}^\infty$, $y_k^+ = \left(\mathbf{v}_k^+; \eta_k^+\right) \in \mathbf{J}_{0,S}(\Omega) \oplus H = \mathcal{H}$, such that the set of projections $\left\{\mathbf{v}_k^+\right\}_{k=1}^\infty$ on $\mathbf{J}_{0,S}(\Omega)$ forms a Riesz basis with finite defect in the space $\mathbf{J}_{0,S}(\Omega)$. This Riesz basis is a p_0–basis (with finite defect) in $\mathbf{J}_{0,S}(\Omega)$ for*

$$p_0 > 6/7 . \tag{4.13}$$

P r o o f. Any operator and its inverse have mutually inverse eigenvalues and the same eigenelements; therefore it is sufficient to verify the existence of a countable set of positive eigenvalues of finite–multiplicity with a unique accumulation point $\mu = 0$ and with the required properties for the eigenelements for $\mathcal{A}^{-1}$. To see this, we consider the maximal nonnegative subspace $\mathcal{L}_+ \subset \mathcal{H}$ introduced in the proof of Lemma 4.9. The subspace $\mathcal{L}_+$ is invariant with respect to the operator $\mathcal{A}^{-1}$ and the angular operator $K = K_+$ corresponds to it (see (4.11), (4.12)). We prove that the restriction $\mathcal{A}^{-1}\big|_{\mathcal{L}_+}$ of the operator $\mathcal{A}^{-1}$ on the subspace $\mathcal{L}_+$ is a compact operator acting in $\mathcal{L}_+$. Indeed, by (4.9), the operator $\mathcal{A}^{-1}\big|_{\mathcal{L}_+}$ acts on any element $y = (\mathbf{v}; K\mathbf{v}) \in \mathcal{L}_+$ according to the law

$$\mathcal{A}^{-1}\big|_{\mathcal{L}_+} y = \begin{pmatrix} \left(\mathcal{A}_{11}^{(-1)} + \mathcal{A}_{12}^{(-1)} K\right) \mathbf{v} \\ \left(\mathcal{A}_{21}^{(-1)} + \mathcal{A}_{22}^{(-1)} K\right) \mathbf{v} \end{pmatrix} . \tag{4.14}$$

Let $P_+|_{\mathcal{L}_+}$ be the restriction on $\mathcal{L}_+$ of the orthoprojector P_+ acting by the law $P_+y := \mathbf{v}$ for all $y = (\mathbf{v};\eta) \in \mathcal{H}$. Since $\mathcal{L}_+$ is invariant with respect to the operator $\mathcal{A}^{-1}$, we obtain from (4.14) that

$$P_+|_{\mathcal{L}_+}\mathcal{A}^{-1}|_{\mathcal{L}_+} = P_+\mathcal{A}^{-1}|_{\mathcal{L}_+} = \left(\mathcal{A}_{11}^{(-1)} + \mathcal{A}_{12}^{(-1)}K\right)P_+|_{\mathcal{L}_+}. \tag{4.15}$$

Since the operator $P_+|_{\mathcal{L}_+}$ maps the subspace $\mathcal{L}_+$ homeomorphic onto $\mathcal{K}^+ = \mathbf{J}_{0,S}(\Omega)$ (see, for instance, [5, p. 37]), we conclude from (4.15) that the operator $\mathcal{A}^{-1}|_{\mathcal{L}_+}$ is similar to the operator $\mathcal{A}_{11}^{(-1)} + \mathcal{A}_{12}^{(-1)}K$, strictly speaking

$$\mathcal{A}^{-1}|_{\mathcal{L}_+} = \left(P_+|_{\mathcal{L}_+}\right)^{-1}\left(\mathcal{A}_{11}^{(-1)} + \mathcal{A}_{12}^{(-1)}K\right)P_+|_{\mathcal{L}_+}. \tag{4.16}$$

Since the angular operator K is bounded and the operators $\mathcal{A}_{11}^{(-1)}$ and $\mathcal{A}_{12}^{(-1)}$ are compact, the operator $\mathcal{A}_{11}^{(-1)} + \mathcal{A}_{12}^{(-1)}K$ is compact. Therefore the operator $\mathcal{A}^{-1}|_{\mathcal{L}_+}$ is also compact, and $\mathcal{L}_+$ is a nonnegative invariant subspace of the class h^+. This fact yields that neutral elements of the subspace $\mathcal{L}_+$ form a finite–dimensional linear subspace $\mathcal{L}_+^0$, which is also invariant with respect to the operator $\mathcal{A}^{-1}$. Let $\sigma(\mathcal{A}^{-1}|_{\mathcal{L}_+}) = \{\nu_k\}_{k=1}^{\infty}$. Then the subspace $\mathcal{L}_+$ can be represented as a direct sum of two invariant subspaces with respect to $\mathcal{A}^{-1}$ in the following way:

$$\mathcal{L}_+ = \operatorname{span}\left\{\mathcal{L}_\nu(\mathcal{A}^{-1}|_{\mathcal{L}_+}) \,:\, \nu \in \sigma\left(\mathcal{A}^{-1}|_{\mathcal{L}_+^0}\right)\right\} \dotplus \mathcal{L},$$
$$\sigma(\mathcal{A}^{-1}|_{\mathcal{L}}) = \sigma(\mathcal{A}^{-1}|_{\mathcal{L}_+}) \setminus \sigma\left(\mathcal{A}^{-1}|_{\mathcal{L}_+^0}\right).$$

Here $\mathcal{L}_\nu(\mathcal{A}^{-1}|_{\mathcal{L}_+})$ denotes the root subspace of the operator $\mathcal{A}^{-1}|_{\mathcal{L}_+}$ corresponding to its eigenvalue ν and the symbol span stands for the linear hull. Since

$$\mathcal{L}_+^0 \subset \operatorname{span}\left\{\mathcal{L}_\nu(\mathcal{A}^{-1}|_{\mathcal{L}_+}) \,:\, \nu \in \sigma\left(\mathcal{A}^{-1}|_{\mathcal{L}_+^0}\right)\right\},$$

the space $\mathcal{L}$ is a Hilbert space with scalar product induced by the indefinite metric. It has finite codimension

$$\dim \operatorname{span}\left\{\mathcal{L}_\nu(\mathcal{A}^{-1}|_{\mathcal{L}_+}) \,:\, \nu \in \sigma\left(\mathcal{A}^{-1}|_{\mathcal{L}_+^0}\right)\right\}$$

in the subspace $\mathcal{L}_+$. Hence there exists a Riesz basis $\{y_n^+\}_{n=1}^{\infty}$, $y_n^+ = (\mathbf{v}_n^+;\eta_n^+) \in \mathbf{J}_{0,S}(\Omega) \oplus H = \mathcal{H}$ consisting of eigenelements of the operator $\mathcal{A}^{-1}|_{\mathcal{L}}$ in $\mathcal{L}$ which has the same finite codimension in the subspace $\mathcal{L}_+$. The projections $\{\mathbf{v}_n^+\}_{n=1}^{\infty}$ onto $\mathbf{J}_{0,S}(\Omega)$ of the basis also form a Riesz basis with the same finite defect in the space $\mathbf{J}_{0,S}(\Omega)$ because the operator $P_+|_{\mathcal{L}_+}$ maps $\mathcal{L}_+$ onto $\mathbf{J}_{0,S}(\Omega)$ homeomorphically.

We now prove that the above–mentioned Riesz basis is a p_0–basis for $p_0 > 6/7$. To this end we consider the equation (4.12) for the angular operator $K = K_+$. The coefficients $\mathcal{A}_{ij}^{(-1)}$ in the right–hand side of the equation are compact operators of some Schatten classes $\mathfrak{S}_p$. Indeed, by (4.9), the operator $\mathcal{A}_{11}^{(-1)}$ belongs to the same class $\mathfrak{S}_p$ as the operator A_a^{-1}, i.e., the operator A^{-1} since $A_a^{-1} = (A + aI)^{-1} = A^{-1}(I + aA^{-1})^{-1}$. According to the asymptotic formula (3.22), we know that $A^{-1} \in \mathfrak{S}_p$ for $p > p_{11} = 3/2$. For the operators $\mathcal{A}_{12}^{(-1)}$ and $\mathcal{A}_{21}^{(-1)}$ we obtain from (4.9) that the corresponding number

$$p = p\left(\mathcal{A}_{12}^{(-1)}\right) = p\left(\mathcal{A}_{21}^{(-1)}\right) = p\left(A_a^{-\frac{1}{2}}Q^*\right) = p(A^{-\frac{1}{2}}Q^*) = p(QA^{-\frac{1}{2}}).$$

The equation (3.45) yields $p(Q) = p(\gamma_n A^{-\frac{1}{2}}) = p(Q^*) = p(\widetilde{Q})$ and we conclude $p(\widetilde{Q}) = p(\widetilde{Q}^*) > 4$ from the asymptotic formula (3.46). Since $p(A^{-\frac{1}{2}}) > 3$, the operators $\mathcal{A}_{12}^{(-1)}$ and $\mathcal{A}_{21}^{(-1)}$ belong to the class $\mathfrak{S}_p$ for $p > \tilde{p}_0$, $1/\tilde{p}_0 = 1/3 + 1/4 = 7/12$. Therefore all terms in the right side of the equation (4.12) are operators of the class $\mathfrak{S}_p$ for $p > 12/7$. Since the operator $\left(\mathcal{A}_{22}^{(-1)}\right)$ is bounded, $K = K_+ \in \mathfrak{S}_p$ for $p > 12/7$.

Thus, according to the assertion in [6, Chap. 4 §3], the elements $\{y_k^+\}_{k=1}^{\infty}$ form a p–basis in the subspace $\mathcal{L}$ for $p > 12/7$, and then, according to the assertion [5, p. 55], the projections of the elements $\{y_k^+\}_{k=1}^{\infty}$ onto $\mathbf{J}_{0,S}(\Omega)$, i.e., the set $\{\mathbf{v}_k^+\}_{k=1}^{\infty}$ forms a p_0–basis in its closed hull for $p_0 > 6/7$. This proves the Theorem. □

Theorem 4.13 *The problem* (4.2) *has a countable set of positive eigenvalues* $\{\lambda_k^-\}_{k=1}^\infty$, $\lambda_k^- > a > 0$, *of finite–multiplicity, with single accumulation point* $\lambda = a$ *and eigenelements* $\{y_k^-\}_{k=1}^\infty$, $y_k^- = (\mathbf{v}_k^-;\eta_k^-) \in \mathbf{J}_{0,S}(\Omega) \oplus H = \mathcal{H}$, *such that the set of projections* $\{\eta_k^-\}_{k=1}^\infty$ *onto* $H = L_2(\Gamma) \ominus \{1_\Gamma\}$ *forms a Riesz basis with finite defect in the space* H. *This Riesz basis is a* p_0*–basis* (*with finite defect*) *in* H *for* $p_0 > 6/7$.

P r o o f. The proof is carried out in the same way as that of Theorem 4.12, but here for the maximal nonpositive subspace $\mathcal{L}_-$, which is invariant with respect to the operator $\mathcal{A}^{-1}$. For the corresponding angular operator K_- we have $K_- = K_+^* \in \mathfrak{S}_p$ ($p > 12/7$), and therefore $\mathcal{L}_- \in h^-$. Further, instead of relation (4.16) we now obtain

$$\mathcal{A}^{-1}\big|_{\mathcal{L}_-} = \big(P_-|_{\mathcal{L}_-}\big)^{-1}\Big(\mathcal{A}_{21}^{(-1)}K_- + \mathcal{A}_{22}^{(-1)}\Big)\,P_-\big|_{\mathcal{L}_-}. \tag{4.17}$$

Hence, the restriction $\mathcal{A}^{-1}|_{\mathcal{L}_-}$ of the operator $\mathcal{A}^{-1}$ to the subspace $\mathcal{L}_-$ is similar to the operator $\mathcal{A}_{21}^{(-1)}K_- + \mathcal{A}_{22}^{(-1)}$. Notice that $\mathcal{A}_{21}^{(-1)} \in \mathfrak{S}_\infty$, $\|K_-\| \le 1$ and, as it is clear from (4.9), the operator $\mathcal{A}_{22}^{(-1)}$, has the following structure:

$$\mathcal{A}_{22}^{(-1)} = a^{-1}\big(I + a^{-1}QQ^*\big)^{-1} =: a^{-1}I + S_{22}\,, \quad S_{22} = S_{22}^* \in \mathfrak{S}_\infty\,. \tag{4.18}$$

Clearly, $S_{22} = S_{22}^*$ and since the operators Q and Q^* are compact (see Lemma 3.13) we obtain

$$\begin{aligned} S_{22} &= a^{-1}\big(I + a^{-1}QQ^*\big)^{-1} - a^{-1}I \\ &= a^{-1}\big(I + a^{-1}QQ^*\big)^{-1}\big[I - \big(I + a^{-1}QQ^*\big)\big] \\ &= -a^{-2}\big(I + a^{-1}QQ^*\big)^{-1}QQ^* \in \mathfrak{S}_\infty\,. \end{aligned}$$

Formula (4.18) yields that we can apply to the operator $\mathcal{A}^{-1}|_{\mathcal{L}_-}$ the same arguments as to the operator $\mathcal{A}^{-1}|_{\mathcal{L}_+}$ in Theorem 4.12. But as a difference to Theorem 4.12 here the sequence of positive eigenvalues $\{\mu_k^-\}_{k=1}^\infty$ corresponding to a Riesz basis $\{y_k^-\}_{k=1}^\infty$ in a new subspace $\mathcal{L} \subset \mathcal{L}_-$, $\mathcal{L}_- = \mathcal{L} \dot{+} \mathcal{L}_-^0$, has the limit point $a^{-1} > 0$. The further arguments of the proof can be fully repeated. ☐

As a corollary of Theorems 4.12 and 4.13, we have the following result.

Theorem 4.14 *The following assertions for the operator* $\mathcal{A}$ *from problem* (4.2) *hold:*

1°. $\dim(\mathcal{F}(\mathcal{A})/\mathcal{F}_0(\mathcal{A})) < \infty$.

2°. $\mathcal{H} = \mathcal{F}(\mathcal{A})$, *i.e., the closure of the linear hull of the root elements of the operator* $\mathcal{A}$ *coincides with the whole* $\mathcal{H} = \mathbf{J}_{0,S}(\Omega) \oplus H$.

3°. $\mathcal{H} = \mathcal{F}_0(\mathcal{A})$ *iff there are no adjoint elements corresponding to the nonreal eigenvalues* λ *of the operator* $\mathcal{A}$, *i.e.,* $\mathcal{L}_\lambda(\mathcal{A}) = \mathrm{Ker}\,(\mathcal{A} - \lambda\mathcal{I})$. (*We recall that the operator* $\mathcal{A}$ *cannot have more than a finite number of nonreal eigenvalues.*)

4°. *There exists an almost* $\mathcal{J}$*–orthonormal Riesz basis formed by the root elements of the operator* $\mathcal{A}$. *If* $\mathcal{H} = \mathcal{F}_0(\mathcal{A})$, *then there exists an almost* $\mathcal{J}$*–orthonormal Riesz basis formed by the eigenelements of* $\mathcal{A}$.

5°. *If* $\mathcal{F}_0(\mathcal{A}) = \mathcal{H}$, *then a* $\mathcal{J}$*–orthonormal basis in the space* $\mathcal{H}$ *formed by the eigenelements of the operator* $\mathcal{A}$ *exists iff* $\sigma(\mathcal{A}) \subset \mathbb{R}$.

6°. *The above–mentioned bases can be chosen as* p*–bases for* $p > 12/7$.

P r o o f. Note that the proof is based on the Theorem 4.7.

1°. We know that $\mathcal{A}$ is a $\mathcal{J}$–selfadjoint operator of the class (H), and its spectrum $\sigma(\mathcal{A})$ has two accumulation points: $a > 0$ and $+\infty$. Therefore, by assertion (b) of Theorem 4.7, we have assertion 1° of the theorem.

2°. Further, for the operator $\mathcal{A}$ the set $s^0(\mathcal{A}) = \emptyset$. Indeed, real points $\lambda \neq a$ can be only eigenvalues of finite multiplicity and therefore have nondegenerated kernels $\mathrm{Ker}\,(\mathcal{A} - \lambda\mathcal{I})$, and the point $\lambda = a$ is not an eigenvalue of the operator $\mathcal{A}$ (Lemma 4.1). These facts yield that the set $\mathrm{span}\,\{\mathcal{L}_\lambda(\mathcal{A})\colon\ \lambda \in s^0(\mathcal{A})\} = \emptyset$, i.e., it is a nondegenerate subspace. According to assertion (d) of Theorem 4.7, the assertion 2° of the theorem is valid.

3°. Since $s^0(\mathcal{A}) = \emptyset$, the assertion (c) of Theorem 4.7 yields the assertion 3°.

4° – 5°. These assertions are the assertions (e) and (f) of Theorem 4.7.

6°. As it was shown in the proofs of Theorem 4.12 and Theorem 4.13, the angular operators $K_\pm$ of the maximal invariant subspaces $\mathcal{L}_\pm$ of the operator $\mathcal{A}^{-1}$ (and of the operator $\mathcal{A}$) belong to the class $\mathfrak{S}_p$ for $p > 12/7$. Therefore, by assertion (g) of Theorem 4.7, assertion 6° holds. ☐

4.3 The connection between the solutions of two spectral problems

In this section we prove that the solutions of the spectral problem (4.2) are directly connected with the solutions of the problem in the form (1.1) for the S. Krein's pencil.

Theorem 4.15 *Let λ_0 be an eigenvalue of the operator $\mathcal{A}$ and y_0, y_1, ..., y_k a chain of eigen– and associated vectors to it, $y_j = (\mathbf{v}_j;\eta_j)$, $j = \overline{0,k}$. Then $\varphi_0, \varphi_1, \ldots, \varphi_k$, $\varphi_j = A^{\frac{1}{2}}\mathbf{v}_j$, form a chain of eigen– and associated vectors to the eigenvalue $\lambda_0 - a$ (in the sense of M. V. Keldysh) for the S. Krein's operator pencil*

$$L(\lambda) := I - \lambda A^{-1} - \lambda^{-1}B, \quad B := \widetilde{Q}^*\widetilde{Q}, \quad \widetilde{Q} = \gamma_n A^{-\frac{1}{2}}. \tag{4.19}$$

Conversely, to each chain φ_0, φ_1, ..., φ_k of eigen– and associated vectors of the pencil (4.19) *to the eigenvalue $\lambda_0 - a$ there corresponds a chain of root elements y_0, y_1, ..., y_k of the operator $\mathcal{A}$ to the eigenvalue λ_0, where*

$$y_j = \begin{pmatrix} A^{-\frac{1}{2}}\varphi_j \\ \widetilde{Q}\sum_{i=0}^{j}(a-\lambda_0)^{i-j-1}\varphi_i \end{pmatrix}, \quad j = \overline{0,k}. \tag{4.20}$$

P r o o f. Let $y_0 = (\mathbf{v}_0;\eta_0)$ be an eigenelement of the operator $\mathcal{A}$ to the eigenvalue λ_0. We conclude from formula (3.52) that

$$\begin{aligned} A_a^{\frac{1}{2}}\left(A_a^{\frac{1}{2}}\mathbf{v}_0 + Q^*\eta_0\right) - \lambda_0\mathbf{v}_0 &= \mathbf{0}, \\ -QA_a^{\frac{1}{2}}\mathbf{v}_0 + a\eta_0 - \lambda_0\eta_0 &= 0. \end{aligned} \tag{4.21}$$

Lemma 4.1 yields $\lambda_0 \neq a$ and we obtain

$$\eta_0 = (a-\lambda_0)^{-1}QA_a^{\frac{1}{2}}\mathbf{v}_0 = (a-\lambda_0)^{-1}\gamma_n\mathbf{v}_0 \tag{4.22}$$

since $Q = G^*A^{-\frac{1}{2}} = \gamma_n A^{-\frac{1}{2}}$. For $y_0 = (\mathbf{v}_0;\eta_0) \in \mathcal{D}(\mathcal{A})$ we know that $\mathbf{v}_0 \in \mathcal{D}(A^{\frac{1}{2}})$ (see (3.51)) and, hence, $\mathbf{v}_0 = A^{-\frac{1}{2}}\varphi_0$ with $\varphi_0 \in \mathbf{J}_{0,S}(\Omega)$. The equations (4.21) and (4.22) yield

$$\begin{aligned} &\eta_0 = (a-\lambda_0)^{-1}\widetilde{Q}\varphi_0, \\ &A_a^{\frac{1}{2}}A^{-\frac{1}{2}}\varphi_0 + (a-\lambda_0)^{-1}Q^*\widetilde{Q}\varphi_0 - \lambda_0 A_a^{-\frac{1}{2}}A^{-\frac{1}{2}}\varphi_0 = \mathbf{0}. \end{aligned} \tag{4.23}$$

We can rewrite the second equation in (4.23) in the form

$$\left(A_a^{\frac{1}{2}}A^{-\frac{1}{2}} - aA_a^{-\frac{1}{2}}A^{-\frac{1}{2}}\right)\varphi_0 - (\lambda_0-a)A_a^{-\frac{1}{2}}A^{-\frac{1}{2}}\varphi_0 - (\lambda_0-a)^{-1}Q^*\widetilde{Q}\varphi_0 = \mathbf{0}. \tag{4.24}$$

Here the operator

$$T_a := A_a^{\frac{1}{2}}A^{-\frac{1}{2}} - aA_a^{-\frac{1}{2}}A^{-\frac{1}{2}} = AA_a^{-\frac{1}{2}}A^{-\frac{1}{2}} = A^{\frac{1}{2}}A_a^{-\frac{1}{2}} \tag{4.25}$$

is bounded and has a bounded inverse $T_a^{-1} = A_a^{\frac{1}{2}}A^{-\frac{1}{2}}$. Further,

$$Q^*\widetilde{Q}\varphi_0 = Q^*\gamma_n A^{-\frac{1}{2}}\varphi_0 = Q^+\gamma_n A^{-\frac{1}{2}}\varphi_0 = A_a^{-\frac{1}{2}}G\widetilde{Q}\varphi_0,$$

since, by Lemma 3.13, we have $\gamma_n A^{-\frac{1}{2}}\varphi_0 \in H_\Gamma^{\frac{1}{2}} = \mathcal{D}(G)$ for any $\varphi_0 \in \mathbf{J}_{0,S}(\Omega)$. Applying the operator T_a^{-1} to both sides of the equation (4.24), we obtain

$$L(\lambda_0 - a)\varphi_0 := \left[I - (\lambda_0-a)A^{-1} - (\lambda_0-a)^{-1}B\right]\varphi_0 = 0 \tag{4.26}$$

since

$$\begin{aligned} T_a^{-1}(\lambda_0 - a)^{-1} Q^* \widetilde{Q}\varphi_0 &= (\lambda_0 - a)^{-1} A_a^{\frac{1}{2}} A^{-\frac{1}{2}} A_a^{-\frac{1}{2}} G\widetilde{Q}\varphi_0 \\ &= (\lambda_0 - a)^{-1} A^{-\frac{1}{2}} G\widetilde{Q}\varphi_0 \\ &= (\lambda_0 - a)^{-1} \big(G^* A^{-\frac{1}{2}}\big)^* \widetilde{Q}\varphi_0 \\ &= (\lambda_0 - a)^{-1} \widetilde{Q}^* \widetilde{Q}\varphi_0 \,. \end{aligned}$$

The formula (4.26) shows that the element $\boldsymbol{\varphi}_0 = A^{-\frac{1}{2}}\mathbf{v}_0$ is an eigenvector of the S. Krein's pencil (4.19) for the eigenvalue $\lambda_0 - a$. It is not difficult to check that from the problem (4.26) we can return to the problem (4.21).

Similar arguments can be applied for the associated vectors of the problem (4.2). Let y_k be the k–th associated element of the operator $\mathcal{A}$ for the eigenvalue λ_0. Then we have the system of equations

$$\begin{aligned} \mathcal{A}y_0 - \lambda_0 y_0 &= 0\,, \\ \mathcal{A}y_1 - \lambda_0 y_1 - y_0 &= 0\,, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \mathcal{A}y_k - \lambda_0 y_k - y_{k-1} &= 0\,. \end{aligned} \tag{4.27}$$

The last equation (4.27) is equivalent to the system

$$\begin{aligned} A_a^{\frac{1}{2}}\Big(A_a^{\frac{1}{2}}\mathbf{v}_k + Q^*\eta_k\Big) - \lambda_0\mathbf{v}_k - \mathbf{v}_{k-1} &= \mathbf{0}\,, \\ -\,QA_a^{\frac{1}{2}}\mathbf{v}_k + a\eta_k - \lambda_0\eta_k - \eta_{k-1} &= 0\,. \end{aligned} \tag{4.28}$$

From the second equation we conclude that

$$\eta_k = (a - \lambda_0)^{-1}[\gamma_n\mathbf{v}_k + \eta_{k-1}]\,. \tag{4.29}$$

After the substitutions

$$\mathbf{v}_j = A^{-\frac{1}{2}}\boldsymbol{\varphi}_j, \quad j = \overline{0,\,k}\,, \tag{4.30}$$

in system (4.28), taking into account the corresponding relations for $j < k$ and the first equality (4.23), we can deduce by induction from (4.29) that

$$\eta_j = \widetilde{Q}\left(\sum_{i=0}^{j}(a - \lambda_0)^{i-j-1}\boldsymbol{\varphi}_i\right), \quad j = \overline{0,\,k}\,. \tag{4.31}$$

Further, in the first equation (4.28) we use the same transformations which were applied above under the transition from the equation (4.21) to (4.26). By (4.31), as a final result we shall have the equation

$$\begin{aligned} \big(\boldsymbol{\varphi}_k - (\lambda_0 - a)A^{-1}\varphi_k - (\lambda_0 - a)^{-1}B\boldsymbol{\varphi}_k\big) + \big(-A^{-1}\boldsymbol{\varphi}_{k-1} + (\lambda_0 - a)^{-2}B\boldsymbol{\varphi}_{k-1}\big) \\ + B\sum_{j=0}^{k-2}(a - \lambda_0)^{j-k-1}\boldsymbol{\varphi}_j = \mathbf{0}\,, \end{aligned} \tag{4.32}$$

which can be rewritten for the pencil $L(\lambda)$ in the form

$$L(\lambda_0 - a)\boldsymbol{\varphi}_k + L'(\lambda_0 - a)\boldsymbol{\varphi}_{k-1} + \sum_{j=2}^{k}(j!)^{-1}L^{(j)}(\lambda_0 - a)\boldsymbol{\varphi}_{k-j} = 0\,. \tag{4.33}$$

It follows that the elements $\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_k$ form a chain of eigen– and associated elements (in M. V. Keldysh's sense) of the operator pencil $L(\lambda)$ for $\lambda = \lambda_0 - a$. These arguments can be reversed, i.e., we can return from (4.33) to the problem (4.27). The formulae (4.20) follow from (4.30) and (4.31). □

Theorem 4.14 and Theorem 4.15 yield the following assertion.

Theorem 4.16 *For the operator $\mathcal{A}$ the following properties are valid.*

1) *The nonreal eigenvalues of the operator $\mathcal{A}$ and those real ones to which there correspond associated elements are located in the segment*

$$M \;:=\; \Big\{\lambda \in \mathbb{C} \,:\, \operatorname{Re}\lambda - a \geq \big(2\,\|A^{-1}\|\big)^{-1},\; |\lambda - a| \leq 2\,\|B\|\Big\} \tag{4.34}$$

of the complex domain $\mathbb{C}$. To this finite set of eigenvalues there correspond neutral eigenelements of the operator $\mathcal{A}$.

Conversely, eigenvalues corresponding to neutral eigenelements of the operator $\mathcal{A}$ are situated in the segment M and, additionally, the operator $\mathcal{A}$ has associated elements for such real eigenvalues.

2) *The eigenvalues λ_k^+ corresponding to positive eigenelements from the nonnegative invariant subspace $\mathcal{L}_+$ are situated on the interval $(a+2\|B\|,\infty)$. Respectively, the eigenvalues λ_k^- corresponding to negative eigenelements from the nonpositive invariant subspace $\mathcal{L}_-$ are situated on the interval $\big(a, a+(2\,\|A^{-1}\|)^{-1}\big)$.*

3) *If the operators A and B fulfill the condition*

$$4\,\|A^{-1}\|\,\|B\| \;<\; 1\,, \tag{4.35}$$

then the operator $\mathcal{A}$ has no neutral eigenelements and, hence, no nonreal eigenvalues and associated vectors. In this case the union of normalized eigenelements $\{y_k^+\}_{k=1}^{\infty} \subset \mathcal{L}_+$ and $\{y_k^-\}_{k=1}^{\infty} \subset \mathcal{L}_-$ form a $\mathcal{J}$–orthonormal basis in the space $\mathcal{H} = \mathbf{J}_{0,S}(\Omega) \oplus H$.

4) *For the eigenvalues λ_k^+ and λ_k^- the two–side estimates*

$$\lambda_k(A) - 2\,\|B\| \;\leq\; \lambda_k^+ - a \;\leq\; \lambda_k(A), \tag{4.36}$$

$$\lambda_k(B) \;\leq\; \lambda_k^- - a \;\leq\; \lambda_k(B)/\big[1 - 2\,\lambda_k(B)\,\|A^{-1}\|\big] \tag{4.37}$$

are fulfilled for $k \in \mathbb{N}$ and the asymptotic formulas

$$\lambda_k^+ \;=\; \lambda_k(A) + O(1) \;=\; \left(\frac{\operatorname{mes}\Omega}{3\pi^2}\right)^{-2/3} k^{2/3}\big[1+o(1)\big] \quad (k\to\infty)\,, \tag{4.38}$$

$$\lambda_k^- \;=\; a + \lambda_k(B)\big[1+o(1)\big] \;=\; a + \left(\frac{\operatorname{mes}\Gamma}{16\pi}\right)^{\frac12} k^{-\frac12}\big[1+o(1)\big] \quad (k\to\infty) \tag{4.39}$$

hold.

P r o o f. 1) Let λ_0 be a nonreal eigenvalue of the operator $\mathcal{A}$ and $y_0 = (\mathbf{v}_0;\eta_0)$ be a corresponding eigenelement. Then, by Theorem 4.15, the element $\varphi_0 = A^{-\frac12}\mathbf{v}_0$ is an eigenelement of the operator $L(\lambda_0 - a)$ (see (4.26)) and, therefore

$$(\lambda_0 - a)^2\big(A^{-1}\varphi_0,\varphi_0\big) - (\lambda_0 - a)(\varphi_0,\varphi_0) + (B\varphi_0,\varphi_0) \;=\; 0\,. \tag{4.40}$$

Since $\lambda_0 - a$ is nonreal, the inequality

$$(\varphi_0,\varphi_0)^2 - 4\big(A^{-1}\varphi_0,\varphi_0\big)(B\varphi_0,\varphi_0) \;<\; 0 \tag{4.41}$$

holds and we conclude for the root $\lambda_0 - a$ of the quadratic equation (4.40) that

$$\begin{aligned} |\lambda_0 - a|^2 \;&=\; \frac{(\varphi_0,\varphi_0)^2 - \big[(\varphi_0,\varphi_0)^2 - 4\big(A^{-1}\varphi_0,\varphi_0\big)(B\varphi_0,\varphi_0)\big]}{\big[2(A^{-1}\varphi_0,\varphi_0)\big]^2} \\ &=\; \frac{(B\varphi_0,\varphi_0)}{(A^{-1}\varphi_0,\varphi_0)} \\ &<\; \frac{4(B\varphi_0,\varphi_0)^2}{(\varphi_0,\varphi_0)^2} \\ &\leq\; 4\,\|B\|^2\,. \end{aligned}$$

Therefore,

$$|\lambda_0 - a| \ < \ 2\,\|B\|\,. \tag{4.42}$$

Further, from the same equation we obtain

$$\operatorname{Re}\lambda_0 - a \ = \ \operatorname{Re}(\lambda_0 - a) \ = \ \frac{(\varphi_0, \varphi_0)}{2(A^{-1}\varphi_0, \varphi_0)} \ \geq \ \big(2\,\|A^{-1}\|\big)^{-1}\,. \tag{4.43}$$

Let now λ_0 be a real eigenvalue to which an eigenelement y_0 and an associated element y_1 correspond. Then, by Theorem 4.15, the eigenelement φ_0 has an associated element φ_1 (see (4.30)), i.e., the equations

$$\begin{aligned} \big(I - (\lambda_0 - a)A^{-1} - (\lambda_0 - a)^{-1}B\big)\varphi_0 \ &= \ \mathbf{0}\,,\\ \big(I - (\lambda_0 - a)A^{-1} - (\lambda_0 - a)^{-1}B\big)\varphi_1 \ &= \ \big(A^{-1} - (\lambda_0 - a)^{-2}B\big)\varphi_0 \end{aligned} \tag{4.44}$$

hold. Hence, we obtain the equation (4.40) and the relation

$$\big(A^{-1}\varphi_0, \varphi_0\big) - (\lambda_0 - a)^{-2}(B\varphi_0, \varphi_0) \ = \ 0\,, \tag{4.45}$$

since

$$(L(\lambda_0 - a)\varphi_1, \varphi_0) \ = \ (\varphi_1, L(\lambda_0 - a)\varphi_0) \ = \ (\varphi_1, \mathbf{0}) \ = \ 0\,.$$

Then

$$\frac{(\varphi_0, \varphi_0)}{2\big(A^{-1}\varphi_0, \varphi_0\big)} \ = \ \lambda_0 - a \ = \ \frac{2(B\varphi_0, \varphi_0)}{(\varphi_0, \varphi_0)}$$

and, therefore

$$\big(2\,\|A^{-1}\|\big)^{-1} \ \leq \ \lambda_0 - a \ = \ |\lambda_0 - a| \ \leq \ 2\,\|B\|\,,$$

i.e., the inequalities (4.34) are valid for real eigenvalues λ_0 with an associated vector.

To prove the converse we first remark the well–known fact that to nonreal eigenvalues of a $\mathcal{J}$–selfadjoint operator $\mathcal{A}$ there correspond neutral eigenelements. If an eigenelement y_0 corresponds to an eigenvalue $\lambda_0 \in \mathbb{R}$ and has an associated element y_1, then the relation (4.45) holds and therefore

$$(\mathcal{J}y_0, y_0)_{\mathcal{H}} \ = \ \big(A^{-1}\varphi_0, \varphi_0\big) - (\lambda_0 - a)^{-2}(B\varphi_0, \varphi_0) \ = \ 0\,, \tag{4.46}$$

i.e., y_0 is a neutral element.

Assume that y_0 is a neutral eigenelement for the eigenvalue λ_0. If $\lambda_0 \neq \overline{\lambda_0}$, then the inequalities (4.42) and (4.43) hold and hence $\lambda_0 \in M$. If $\lambda_0 \in \mathbb{R}$, then it is sufficient to deduce that the second equation (4.44) has a nontrivial solution φ_1. Since the operator $(\lambda_0 - a)A^{-1} - (\lambda_0 - a)^{-1}B$ is compact and selfadjoint, the assertion is valid iff the right side of (4.44) is orthogonal to the solution of the homogeneous equation, i.e., to $\varphi_0 \neq \mathbf{0}$. Since y_0 is a neutral element, it fulfills the condition (4.46) and we obtain that there exists a solution $\varphi_1 \neq \mathbf{0}$ of the second equation of (4.44). Hence, y_1 is an associated vector to the eigenelement y_0.

2) Let λ_0 be a positive eigenvalue of the operator $\mathcal{A}$ corresponding to a positive eigenelement y_0 from the invariant nonnegative subspace $\mathcal{L}_+$. Then

$$0 \ < \ (\mathcal{J}y_0, y_0)_{\mathcal{H}} \ = \ \big(A^{-1}\varphi_0, \varphi_0\big) - (\lambda_0 - a)^{-2}(B\varphi_0, \varphi_0) \tag{4.47}$$

and the inequality, which is opposite to (4.41), holds. Since $\lambda_0 - a > 0$ we conclude by using formula (4.40) that

$$\begin{aligned} (\varphi_0, \varphi_0) - 2(\lambda_0 - a)\big(A^{-1}\varphi_0, \varphi_0\big) \ &= \ -(\lambda_0 - a)\big(A^{-1}\varphi_0, \varphi_0\big) + (\lambda_0 - a)^{-1}(B\varphi_0, \varphi_0)\\ &= \ -(\lambda_0 - a)\big[\big(A^{-1}\varphi_0, \varphi_0\big) - (\lambda_0 - a)^{-2}(B\varphi_0, \varphi_0)\big]\\ &< \ 0\,. \end{aligned}$$

Therefore,

$$\lambda_0 - a \;>\; \frac{(\varphi_0,\varphi_0)}{2\big(A^{-1}\varphi_0,\varphi_0\big)} \;\geq\; \big(2\,\|A^{-1}\|\big)^{-1}. \tag{4.48}$$

It was proved above, that on the interval $\big[a + (2\,\|A^{-1}\|)^{-1}, a + 2\,\|B\|\big]$ there are only eigenvalues to which neutral eigenelements correspond. Since the element

$$y_0 \;=\; \big(A^{-\frac{1}{2}}\varphi_0; (\lambda_0 - a)^{-1}\widetilde{Q}\varphi_0\big)$$

is positive by assumption, we conclude that $\lambda_0 \in (a + 2\,\|B\|, +\infty)$.

The proof of the second assertion in 2) for eigenvalues λ_k^- that correspond to negative eigenelements $y_k^- \in \mathcal{L}_-$, is carried out in the same way using the inequality

$$(\varphi_0,\varphi_0) - 2(\lambda_0 - a)^{-1}(B\varphi_0,\varphi_0) \;=\; (\lambda_0 - a)\big[\big(A^{-1}\varphi_0,\varphi_0\big) - (\lambda_0 - a)^{-2}(B\varphi_0,\varphi_0)\big] \;<\; 0\,.$$

3) The condition (4.35) yields that the segment $M = \emptyset$. Hence, the operator $\mathcal{A}$ has no nonreal eigenvalues and each of its eigenelements is definite and therefore has no associated elements. Then the invariant subspaces $\mathcal{L}_+$ and $\mathcal{L}_-$ are uniformly definite, and the basis, which is formed by the eigenelements of the operator $\mathcal{A}$, is a $\mathcal{J}$–orthonormal one in the space $\mathcal{H} = \mathbf{J}_{0,S}(\Omega) \oplus H$ (see Theorem 4.14, assertion 5).

4) The estimates (4.36) and (4.37) for the operator pencil (4.19) were deduced in [23, p. 300]. The asymptotic formulas (4.38) and (4.39) follow from the asymptotic formulas (3.22), (3.46) and the estimates (4.36), (4.37). □

4.4 Fourier series

Return to the Cauchy problem (3.55)–(3.57) to which the initial boundary value problem (3.3)–(3.6) on small motions of a viscous fluid was reduced. If the condition (4.35) is valid, then the operator $\mathcal{A}$ has a $\mathcal{J}$–orthonormal basis formed from eigenelements $\{y_k^+\}_{k=1}^\infty \subset \mathcal{L}_+$, $\{y_k^-\}_{k=1}^\infty \subset \mathcal{L}_-$:

$$\big[y_k^+, y_j^+\big] \;=\; \delta_{kj}\,, \quad \big[y_k^+, y_l^-\big] \;=\; 0\,, \quad \big[y_l^-, y_m^-\big] \;=\; -\delta_{lm}\,. \tag{4.49}$$

This allows us to obtain the solution $y(t)$ of the problem (3.55) in the form of a Fourier series.

Represent $y(t)$ in the form

$$y(t) \;=\; \sum_{k=1}^\infty c_k^+(t) y_k^+ + \sum_{l=1}^\infty c_l^-(t) y_l^-\,. \tag{4.50}$$

Using the formulae (4.49), we obtain the system of Cauchy problems:

$$\frac{dc_k^+}{dt}(t) + \lambda_k^+ c_k^+(t) \;=\; f_k^+(t) \;:=\; e^{-at}\left[\begin{pmatrix}\mathbf{f}_{0,S}(t)\\ 0\end{pmatrix}, y_k^+\right], \qquad c_k^+(0) \;=\; [y^0, y_k^+]\,, \qquad k \in \mathbb{N};$$

$$\frac{dc_l^-}{dt}(t) + \lambda_l^- c_l^-(t) \;=\; f_l^-(t) \;:=\; -e^{-at}\left[\begin{pmatrix}\mathbf{f}_{0,S}(t)\\ 0\end{pmatrix}, y_l^-\right], \quad c_l^-(0) \;=\; -[y^0, y_l^-]\,, \quad l \in \mathbb{N}.$$

From this we obtain the formulae for the unknown functions $c_k^+(t)$ and $c_l^-(t)$ and, hence, also the formula for $y(t)$ and for the solution

$$\begin{pmatrix}\mathbf{u}(t)\\ \zeta(t)\end{pmatrix} \;=\; e^{at}\begin{pmatrix}\mathbf{v}(t)\\ \eta(t)\end{pmatrix} \;=\; e^{at} y(t)$$

(see (3.38)). The solution has the form

$$\begin{aligned} \begin{pmatrix} \mathbf{u}(t) \\ \zeta(t) \end{pmatrix} &= e^{at} y(t) \\ &= \sum_{k=1}^{\infty} e^{-(\lambda_k^+ - a)t} \left[\begin{pmatrix} \mathbf{u}^0 \\ \zeta^0 \end{pmatrix}, y_k^+ \right] - \sum_{l=1}^{\infty} e^{-(\lambda_l^- - a)t} \left[\begin{pmatrix} \mathbf{u}^0 \\ \zeta^0 \end{pmatrix}, y_l^- \right] \\ &\quad + \left\{ \sum_{k=1}^{\infty} \int_0^t e^{-(\lambda_k^+ - a)(t-s)} \left[\begin{pmatrix} \mathbf{f}_{0,S}(s) \\ 0 \end{pmatrix}, y_k^+ \right] ds \right. \\ &\quad \left. - \sum_{l=1}^{\infty} \int_0^t e^{-(\lambda_l^- - a)(t-s)} \left[\begin{pmatrix} \mathbf{f}_{0,S}(s) \\ 0 \end{pmatrix}, y_l^- \right] ds \right\} \\ &=: z_+(t) + z_-(t) + z_0(t) \,. \end{aligned} \tag{4.51}$$

Here each term on the right side has a clear physical sense: the function $z_+(t)$ describes the free motions of a viscous fluid connected with interior dissipative waves (rapidly damped aperiodic motions); the function $z_-(t)$ describes the free motions connected with surface waves (slowly damped aperiodic motions); the function $z_0(t)$ gives us the solution of the problem (for zero initial data) due to effect of external forces (forced motions). The solution (4.51) also allows us to extract terms which depend on the initial velocity field $\mathbf{u}^0$ or the initial displacement ζ^0 of the free fluid surface.

4.5 Final remarks

In conclusion of the paper we mention some facts connected directly with the problem.

1. Remember that the initial physical problem contains the parameters $\nu > 0$ (the viscosity of the fluid) and $g > 0$ (the gravity acceleration). In Subsection 3.4 and further we supposed these parameters to be equal to 1. Returning to the initial notations, we must write νA instead of A in all formulas and change G to $g^{\frac{1}{2}}G$. In particular, the condition (4.35) then has the form

$$4g \left\|A^{-1}\right\| \|B\| < \nu^2 \tag{4.52}$$

where $B = \left(\gamma_n A^{-\frac{1}{2}}\right)^* \left(\gamma_n A^{-\frac{1}{2}}\right)$. We conclude that the operator $\mathcal{A}$ has only real eigenvalues for sufficient large viscosity ν and, in this case, its eigenelements form a $\mathcal{J}$–orthogonal basis in the space $\mathcal{H}$. It can be proved (see, for instance, [23, p. 205]), that in this case

$$\lambda_k^+ \in (a + r_+, \infty) \,, \quad \lambda_k^- \in (a, a + r_-)$$

for $k \in \mathbb{N}$ where

$$r_\pm = \frac{\nu \pm \left(\nu^2 - 4g \|A^{-1}\| \|B\|\right)^{\frac{1}{2}}}{2 \|A^{-1}\|} \,, \quad 0 < r_- < r_+ \,.$$

For the branches λ_k^+ and λ_k^- the two–side estimates

$$\nu\lambda_k(A) - 2g\nu^{-1}\|B\| \le \lambda_k^+ - a \le \nu\lambda_k(A) \tag{4.53}$$

and

$$g\nu^{-1}\lambda_k(B) \le \lambda_k^- - a \le g\nu^{-1}\lambda_k(B) / \left[1 - 2g\nu^{-2}\lambda_k(B) \left\|A^{-1}\right\|\right] \tag{4.54}$$

hold for all $k \in \mathbb{N}$. According to (3.22) and (3.46) these branches have the following asymptotic behaviour

$$\lambda_k^+ = \nu\lambda_k(A) + O(1) = \nu \left(\frac{\operatorname{mes}\Omega}{3\pi^2}\right)^{-2/3} k^{2/3} \left[1 + o(1)\right] \qquad (k \to \infty) \tag{4.55}$$

and

$$\lambda_k^- \;=\; a + g\nu^{-1}\lambda_k(B)\big[1+o(1)\big] \;=\; a + g\nu^{-1}\left(\frac{\operatorname{mes}\Gamma}{16\pi}\right)^{\frac{1}{2}} k^{-\frac{1}{2}}\big[1+o(1)\big] \quad (k\to\infty)\,. \tag{4.56}$$

We note that there correspond so–called interior dissipative waves in the viscous fluid to the branch of eigenvalues $\{\lambda_k^+\}$, $k\in\mathbb{N}$, and so–called surface gravitational waves to the branch of eigenvalues $\{\lambda_k^-\}$, $k\in\mathbb{N}$. Their properties were described in Chapter 7 of the monograph [23].

2. It is necessary to take into account physical dimensions of the unknown and known variables under the mathematical consideration of the physical problems. If we introduce typical values for these variables or parameters we use the opportunity to pass to undimensional variables and to consider the initial boundary value problem (3.3) – (3.6) already in undimensional variables. This allows us to use formulae of the form

$$\|y\|_{\mathcal{H}}^2 \;=\; \|\mathbf{v}\|^2 + \|\eta\|_0^2$$

and we do not need to calculate the physical dimensions of the terms.

3. The change of the spectral parameter $\lambda=\mu^{-1}=:\lambda(\mu)$ was done with the transformation of the problem (4.2) to the problem (4.10). It is evident that under such a change both spectral problems (for the operators $\mathcal{A}$ and $\mathcal{A}^{-1}$) have the same eigenelements. But it is not true for associated elements. It can be shown that the associated vectors for the problems (4.2) and (4.10) are connected by a triangle transformation (see, for instance, [23, p. 73]). Let, for example, λ_0 be an eigenvalue and $y_0, y_1, \dots, y_k$ a corresponding chain of an eigenelement and associated vectors of the operator $\mathcal{A}$. Define $\tilde y_0 = y_0$ and

$$\tilde y_k \;=\; (-1)^k \sum_{l_1+2l_2+\ldots+jl_j=k} \frac{(l_1+l_2+\ldots+l_j)!\, y_{l_1+l_2+\ldots+l_j}}{l_1!\,l_2!\,\ldots\,l_j!\,\mu_0^{2l_1+\ldots+(j+1)l_j}}\,,\quad k\in\mathbb{N}. \tag{4.57}$$

In particular,

$$\tilde y_1 \;=\; -\mu_0^{-2}y_1\,,\quad \tilde y_2 \;=\; \mu_0^{-3}y_1+\mu_0{}^{-4}y_2\,,\quad \tilde y_3 \;=\; -\big(\mu_0{}^{-4}y_1+2\mu_0{}^{-5}y_2+\mu_0{}^{-6}y_3\big)\,.$$

Then $\tilde y_0, \tilde y_1, \dots, \tilde y_k$ form a chain of an eigenelement and eigenvectors to the eigenvalue $\mu_0=\lambda_0{}^{-1}$ of the operator $\mathcal{A}^{-1}$ (and vice versa).

4. The approach applied in this paper and based on the theory of operator matrices with unbounded entries can be used also in more complicated hydrodynamic problems than the classical S. Krein problem. In particular, it can be applied to the problem on small motions of a viscous fluid which uniformly rotates in a partially filled container. The fluid can be either a heavy one (ordinary conditions) or a capillary one (conditions of weightlessness).

Let a heavy viscous fluid uniformly rotate with a constant angular velocity $\omega_0=\omega_0\mathbf{e}_3$ in a partially filled container. Then its free surface Γ is a rotation paraboloid and the equilibrium pressure is equal to

$$P_0(x) \;=\; p_a - \rho g x_3 + \frac{1}{2}\rho\omega_0^2\big(x_1^2+x_2^2\big)\,. \tag{4.58}$$

The equations describing the small oscillations near the uniform rotation of the fluid are similar to those in (3.3) – (3.6), but need the following variations (see [23, p. 313]). In the first equation (3.3) we must add the Coriolis term $-2\omega_0\mathbf{u}\times\mathbf{e}_3$ (on the left). Further, in condition (3.5) the ordinary partial derivatives must be changed to corresponding covariant derivatives (in local curvilinear coordinate systems in some vicinity of the free surface Γ), and the term $\rho g\zeta$ in the right side of the last condition (3.5) must be changed by the term $P_\Gamma a(\hat\xi)P_\Gamma\zeta$. Here P_Γ is an orthoprojection onto $H=L_2(\Gamma)\ominus\{1_\Gamma\}$, $\hat\xi:=(\xi_1,\xi_2)$ a point on Γ and $a(\hat\xi):=(\nabla P_0(x)\cdot\mathbf{n})|_\Gamma$ a known function which is continuous and positive on Γ.

If we repeat the scheme of the transition from the initial boundary value problem to the system of operator equations (see Subsections 3.2, 3.3), we obtain the system

$$\begin{aligned}
&\frac{d\mathbf{u}}{dt} + 2i\omega_0 S\mathbf{u} + \nu A\mathbf{u} + GB_0\zeta \;=\; \mathbf{f}_{0,S}\,,\\
&\frac{d\zeta}{dt} - \gamma_n\mathbf{u} \;=\; 0\,,\quad \mathbf{u}(0) \;=\; \mathbf{u}^0\,,\quad \zeta(0) \;=\; \zeta^0\,,
\end{aligned} \tag{4.59}$$

instead of the problem (3.27.) Here $S\mathbf{u} := iP_{0,S}(\mathbf{u} \times \mathbf{e}_3)$, $S = S^*$, $\|S\| = 1$, and $B_0 := P_\Gamma a(\hat{\xi}) P_\Gamma$ is a bounded selfadjoint positive definite operator acting in the space H.

Using the substitution $\eta := B_0^{\frac{1}{2}} \zeta$ and applying the operator $B_0^{\frac{1}{2}}$ to both sides of the second equation of the system (4.59) we obtain the problem

$$\begin{aligned} &\frac{d\mathbf{u}}{dt} + 2i\omega_0 S\mathbf{u} + \nu A\mathbf{u} + GB_0^{\frac{1}{2}}\eta \;=\; \mathbf{f}_{0,S}\,, \\ &\frac{d\eta}{dt} - B_0^{\frac{1}{2}} G^*\mathbf{u} \;=\; 0, \qquad \mathbf{u}(0) \;=\; \mathbf{u}^0, \qquad \eta(0) \;=\; \eta^0 \;:=\; B_0^{\frac{1}{2}}\zeta^0. \end{aligned} \tag{4.60}$$

The corresponding operator matrix

$$\tilde{\mathcal{A}}_0 \;:=\; \begin{pmatrix} \nu A + 2i\omega_0 S & GB_0^{\frac{1}{2}} \\ -B_0^{\frac{1}{2}} G^* & 0 \end{pmatrix} \tag{4.61}$$

is a generalization of the operator $\mathcal{A}_0$ from (3.35). The operator $\tilde{\mathcal{A}}_0$ is also accretive and the closure $\tilde{\mathcal{A}}$ of the operator $\tilde{\mathcal{A}}_a := \tilde{\mathcal{A}}_0 + a\mathcal{I}$, $a > 0$, is maximal uniformly accretive. This property allows us to investigate the problem (4.59), to prove the existence of a unique strong solution and to study the spectral properties of the operator $\tilde{\mathcal{A}}$.

Acknowledgements The research of T. Ya. Azizov was supported by the Grants NWO–RFBR 047–008–008 and 99–01–00391.

V. Hardt, N. Kopachevsky and R. Mennicken were partially supported by the Deutsche Forschungsgemeinschaft (DFG).

V. Hardt is highly indebted to the Institute of Mathematical Analysis of the Simferopol State University and the participants of the "Tenth Crimean Autumn Mathematical School–Symposium on Spectral and Evolutionary Problems" for the kind hospitality during his stay in Crimea in 1999.

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