On Abstract Green's formula for a Triple of Hilbert spaces and applications.

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Let for a triple of Hilbert spaces E, F, G the following conditions be satisfied: the space F is dense embedded in the space E; there exists an operator $\gamma : F \to G_+ \subset G$ (a trace operator) that acts boundary from F on the space G_+ dense embedded in G.

Then there exist a unique operator $L : F \to F^*$ and a unique operator $\partial : F \to (G_+)^*$ such that the following abstract Green's Formula is valid:

$$\langle \eta, Lu \rangle_E = (\eta, u)_F - \langle \gamma\eta, \partial u \rangle_G , \quad \forall \eta, v \in F. \tag{1}$$

If $E = L_2(\Omega)$, $\Omega \subset \mathbb{R}^m$, $\partial\Omega$ is Lipschitzian, $F = H^1(\Omega)$, $G = L_2(\partial\Omega)$, $G_+ = H^{1/2}(\partial\Omega)$, $\gamma u := u|_{\partial\Omega}$ $(\forall u \in H^1(\Omega))$, we have, as a special case of (1), the famous Green's Formula for the Laplace operator:

$$\langle \eta, -\Delta u + u \rangle_{L_2(\Omega)} = \int\limits_\Omega (\nabla\eta \cdot \nabla u + \eta u)\, d\Omega - \left\langle \gamma\eta, \frac{\partial u}{\partial n} \right\rangle_{L_2(\partial\Omega)}, \quad \forall \eta, v \in H^1(\Omega). \tag{2}$$

Let $\Omega \subset \mathbb{R}^3$, $\Gamma \subset \partial\Omega : x_3 = 0$, $\operatorname{mes}\Gamma > 0$, $S := \partial\Omega \backslash \overline{\Gamma}$, $\operatorname{mes} S > 0$. If $E = \vec{J}_{0,S}(\Omega) = \{\vec{u} \in \vec{L}_2(\Omega) : \operatorname{div}\vec{u} = 0\ (in\ \Omega),\ u_n := \vec{u}\cdot\vec{n} = 0\ (on\ S)\}$, $F = \vec{J}^1_{0,S}(\Omega) = \{\vec{u} \in \vec{H}^1(\Omega) : \operatorname{div}\vec{u} = 0\ (in\ \Omega),\ \vec{u} = \vec{0}\ (on\ S)\}$, $G = L_{2,\Gamma} := L_2(\Gamma) \ominus \{1_\Gamma\}$, $\gamma\vec{u} := (\vec{u}\cdot\vec{n})|_\Gamma$, $G_+ = L_{2,\Gamma} \bigcap H^{1/2}(\Gamma)$, then we have from (1) the Green's Formula for the Stokes operator:

$$\langle \vec{\eta}, \nabla p - \Delta\vec{u} + \vec{u} \rangle_{\vec{L}_2(\Omega)} = \int\limits_\Omega \left(\frac{1}{2} \sum_{i,k=1}^{3} \tau_{ik}(\vec{\eta})\tau_{ik}(\vec{u}) + \vec{\eta}\cdot\vec{u} \right) d\Omega - \langle \gamma\vec{\eta}, \tau_{33}(\vec{u}) - p \rangle_{L_2(\Gamma)},$$

$$\tau_{ik}(\vec{u}) := \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}, \quad \forall \vec{\eta}, \vec{u} \in \vec{J}^1_{0,S}(\Omega) : \tau_{i3}(\vec{u}) = 0\ (on\ \Gamma),\ i = 1, 2,\ \forall \nabla p \in (J^1_{0,S}(\Omega))^*.$$

References

[1] Kopachevsky N. D., Krein S. G., Ngo Zuy Kan *Operatornye metody v linejnoj gidrodinamike: Evoljucionnye i spektral'nye zadachi.* – Moskva: Nauka, 1989, – 416 s.

[2] Kopachevsky N. D., Krein S. G. *Abstraktnaja formula Grina dlja trojki gil'bertovyh prostranstv, abstraktnye kraevye i spektral'nye zadachi.* // Ukrainskij matematicheskij vestnik (to appear in 2004), pp.20.

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