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MULTICOMPONENT CONJUGATION PROBLEMS AND AUXILIARY ABSTRACT BOUNDARY-VALUE PROBLEMS

V. I. Voititsky, N. D. Kopachevsky, and P. A. Starkov UDC 517.98

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1. Introduction

Auxiliary abstract boundary-value problems generated by conjugation problems are considered. Conjugation problems are problems in which interrelations between unknown functions defined in adjoining domains are given only through the boundaries of these domains.

1.1. On the abstract operator approach to conjugation problems. The papers of Agranovich and others (see [3–5]) and his lectures at the annual Crimean Autumnal Mathematical School (Laspy–Batiliman) have become the initial boost for the authors.

Conjugation problems containing a spectral parameter on the boundary of conjugation μ and a fixed parameter $\lambda \in \mathbb{C}$ in the equations were studied in these papers. Moreover, the first domain is fixed and the second domain is a complement to the whole space. In addition, conditions of emission or decay of a solution at infinity are imposed.

Such problems arise in the diffraction theory (see [4]), where the second domain can be bounded; then the Dirichlet condition or any homogeneous condition is imposed. Note that the case where

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the spectral parameter of the equations is $\lambda \in \mathbb{C}$ and the fixed parameter at the boundary of the conjugation of the domains is $\mu \in \mathbb{C}$ was also discussed in [4].

The main method used in [4, 5], where μ is a spectral parameter, is the reduction of the problem to an integral operator equation for functions given at the boundary of the conjugation. Here the parameter λ was included in the kernels of the corresponding integral operators. This can make the study of such operators difficult.

In [52] and other papers of this author (see [53–55]), another method is used. It allows one to study a problem with fixed λ and the spectral parameter μ and an inverse variant simultaneously. Here the method of bilinear forms connected to the problem and of corresponding operators and equipotential technique based on introducing the operators of auxiliary boundary-value problems are used.

The method of operators of auxiliary problems was, seemingly, first used by S. G. Krein. In particular, in monographs [28, 29, 31], it was used for the reduction of the initial-value problem to a spectral problem for a operator sheaf (i.e., an operator-value function depending on a spectral parameter) acting on some Hilbert space.

The present paper considers the abstract auxiliary boundary-value problems that generalize similar conjugation problems. We assume that there are several adjoining domains and the condition of conjugation is imposed on the parts of the general boundaries of these domains. Such problems are quite frequent. Here the corresponding generalizations of diffraction problems, the Stefan spectral problem with the Gibbs–Thompson condition, and the Krein problem about the normal oscillations of a heavy viscous liquid in an open container are considered in detail. The same method can be used for the problem of motion of dynamical systems where the energy dissipates at the surface.

1.2. On the history of the abstract Green formula. Let Ω be an arbitrary domain in $\mathbb{R}^m$ with a boundary $\Gamma := \partial\Omega$. We know that the Green formula

$$\int_\Omega \eta(u - \Delta u)\, d\Omega = \int_\Omega [\nabla\eta \cdot \nabla u + \eta u]\, d\Omega - \int_\Gamma \eta \frac{\partial u}{\partial n}\, d\Gamma, \quad \Delta u := \sum_{k=1}^{m} \frac{\partial^2 u}{\partial x_k^2} \tag{1.1}$$

is valid for a twice continuously differentiable function $u = u(x)$, $x \in \Omega$, a continuously differentiable function $\eta = \eta(x)$, and a sufficiently smooth boundary $\Gamma = \partial\Omega$. We can rewrite this formula in the following form:

$$(\eta, Lu)_{L_2(\Omega)} = (\eta, u)_{H^1(\Omega)} - \left(\gamma\eta, \frac{\partial u}{\partial n}\right)_{L_2(\Gamma)}, \tag{1.2}$$

$$Lu := u - \Delta u, \quad \gamma\eta := \eta\big|_\Gamma. \tag{1.3}$$

Here γ is a trace operator, $\partial/\partial n$ is an outward normal to Γ derivative, and $L_2(\Omega), H^1(\Omega)$, and $L_2(\Gamma)$ are the standard functional Hilbert spaces with the corresponding norms.

The Green formula (1.2) (the first Green formula for the Laplace operator) can be generalized in several ways. First, we can take abstract Hilbert spaces E, F, and G that satisfy some connection conditions instead of concrete Hilbert spaces $L_2(\Omega)$, $H^1(\Omega)$, and $L_2(\Gamma)$. Second, in formula (1.2), instead of the inner product in the first and last terms, we can take their continuous extensions that are functionals (see below). Third, in (1.2) the boundary $\Gamma = \partial\Omega$ can be Lipschitzian.

The following facts are valid.

Theorem 1.1. *Let the following conditions for the triplet of abstract Hilbert spaces $\{E, (\cdot,\cdot)_E\}$, $\{F, (\cdot,\cdot)_F\}$, and $\{G, (\cdot,\cdot)_G\}$ with inner products and for an operator γ, which is further called the trace operator, hold* (*see* [25, 30]):

1°. *The space F is embedded continuously and compactly in the space E* (*the notation is $F \hookrightarrow E$*), *i.e., F is compact in E and there exists a constant $a > 0$ such that*

$$\|u\|_E \le a\|u\|_F \quad \forall u \in F. \tag{1.4}$$

2°. *The trace operator γ is bounded and acts from F onto the space $G_+ \hookrightarrow G$ and*

$$\|\gamma u\|_G \le b\|u\|_F \quad \forall u \in F,\ b > 0. \tag{1.5}$$

Then there exist operators $L : F \to F^*$ and $\partial : F \to (G_+)^*$ uniquely defined by E, F, and G (with the inner products) and γ such that the following abstract Green formula holds:

$$\langle \eta, Lu\rangle_E = (\eta, u)_F - \langle \gamma\eta, \partial u\rangle_G \quad \forall \eta, u \in F. \tag{1.6}$$

Here the values of the functionals $Lu \in F^*$ and $\partial u \in (G_+)^*$ on the elements $\eta \in F$ and $\gamma\eta \in G_+$ respectively are denoted by the oblique diagonal braces.

If the conditions

$$E = L_2(\Omega),\ F = H^1(\Omega),\ G = L_2(\Gamma),\ \Gamma = \partial\Omega,\ \gamma\eta := \eta\big|_\Gamma \tag{1.7}$$

hold, then the following fact (see, e.g., [25]) follows from the Gagliardo theorem (see [18]) and Sec. 2.2.

Theorem 1.2. *The following Green formula holds for the domain $\Omega \subset \mathbb{R}^m$ with the Lipschitz boundary $\Gamma = \partial\Omega$:*

$$\langle \eta, Lu\rangle_{L_2(\Omega)} = (\eta, u)_{H^1(\Omega)} - \left\langle \gamma\eta, \frac{\partial u}{\partial n}\right\rangle_{L_2(\Gamma)} \quad \forall \eta, u \in H^1(\Omega), \tag{1.8}$$

$$Lu := u - \Delta u \in (H^1(\Omega))^*, \quad \frac{\partial u}{\partial n}\Big|_\Gamma \in (H^{1/2}(\Gamma))^* = H^{-1/2}(\Gamma). \tag{1.9}$$

Hence, the Green formula (1.8) generalizes formulas (1.1) and (1.2) for the less smooth functions and the Lipschitz boundary $\Gamma = \partial\Omega$.

Theorems 1.1 and 1.2 will be actively used below and will be generalized to the case of conjugation problems, where the unknown functions given at different domains satisfy some conditions of conjugation for the parts of the boundaries of adjoining domains.

Let us say some words about the history of abstract Green formulas. In [31], it was assumed that $\ker\gamma =: N$ is compact in E and the following formula was obtained:

$$(\eta, Lu)_E = (\eta, u)_F - (\gamma\eta, \partial u)_G \quad \forall \eta \in F,\ \forall u \in \mathcal{D}(L) \subset F \subset E. \tag{1.10}$$

S. Krein thought that he was the first who proved it (see [31, p. 119]). Along with formula (1.10), the abstract scheme of the study of boundary-value problems was obtained (see [31, Sec. 1.3]).

The property $\overline{N} = E$ is well known for the triple of spaces (1.7) as $H^1_0(\Omega) = \ker\gamma = N$ is dense in $L_2(\Omega) = E$.

The further strengthening of the Green abstract formula (1.10) can be found in [25, 30]. In particular, the condition

$$\overline{N} = E \tag{1.11}$$

was removed in [25]. It was ascertained later that the Green abstract formula had been proved by Oben (see [46, Chap. 6] and the reference to the original paper of 1970). However, we see bilinear F-coercive form instead of the inner product (see (1.10)) there and E contains the actual range of the abstract differential expression L. Moreover, conditions (1.11), 1°, and 2° and Theorem 1.1 were used.

Finally, the abstract Green formula in the form of Oben but without the reference to [46] was used in the monograph of Showalter (see [50]). Note that the abstract Green formula in slightly different form (according to the second Green formula for the Laplace operator) is given in [40, p. 58].

1.3. On the results of the paper. Let us present the main results of this paper. In Sec. 2, we obtain the sufficient conditions (see Lemmas 2.1 and 2.2) for the Green formula to have the following form:

$$\langle \eta, Lu\rangle_E = (\eta, u)_F - \sum_{k=1}^{q} \langle \gamma_k\eta, \partial_k u\rangle_G \quad \forall \eta, u \in F, \tag{1.12}$$

where γ_k is an abstract trace operator to the part of the boundary of the domain and ∂_k is an abstract analog of the outward normal derivative. It is proved (Theorem 2.2) that this formula is valid for a domain $\Omega \subset \mathbb{R}^m$ with a Lipschitz boundary $\partial\Omega$ (see (2.51)). In Sec. 2.3, the multicomponent conjugation problems in diffraction theory and their abstract analogs are formulated.

In Sec. 3, the auxiliary abstract boundary-value problems (problems of S. G. Krein) and the operators of such problems are studied. These problems can be easily formulated by using the abstract Green formula for multicomponent conjugation problems (see (3.29), (3.31)–(3.33), and (3.35)) and the representation theorem for any element of the space of solutions (Theorem 3.3).

In Sec. 4, the main equations arising in applications are derived. For the spectral Stefan problem, we obtain the equation in the following form (see (4.12)):

$$\eta = \lambda\left(\mathcal{A}\eta + \mathcal{B}\eta\right), \tag{1.13}$$

where $\mathcal{A}$ is a compact positive operator and $\mathcal{B}$ is a compact nonnegative operator, which are generated by the first and second auxiliary abstract boundary-value problems. The following spectral problem arises for the problems of the diffraction theory:

$$\eta + \lambda\mathcal{A}\eta - \mu\mathcal{B}\eta = 0, \tag{1.14}$$

where either λ or μ is a spectral parameter and the other is fixed. The following equation arises in problems of the type of S. G. Krein:

$$\eta = \lambda\mathcal{A}\eta + \lambda^{-1}\mathcal{B}\eta. \tag{1.15}$$

The general properties of operator coefficients $\mathcal{A}$ and $\mathcal{B}$ are the same. The problem of a bounded self-adjoint operator (see Sec. 4.4) is reduced to the problem (see (4.62))

$$\xi = \lambda A^{-1}\xi - \lambda^{-1}B\xi, \quad 0 < A^{-1} \in \mathfrak{S}_\infty, \quad 0 \leq B \in \mathfrak{S}_\infty. \tag{1.16}$$

The spectral problem of motion of dynamical systems, where the energy dissipates at the surface, leads to the following equation:

$$\eta - \lambda\beta\mathcal{B}\eta + \lambda^2\mathcal{A}\eta = 0, \quad \beta > 0, \tag{1.17}$$

where the coefficients are the same as in (1.13)–(1.15).

The general properties of every problem (1.13)–(1.16) are given in Sec. 4 as theorems and lemmas.

The authors dedicate this paper to the memory of the eminent mathematician and outstanding human being L. R. Volevich.

2. The Abstract Green Formula for Conjugation Problems

In this section, the abstract Green formula is derived for the triple of Hilbert spaces, where it is easy to study mixed boundary-value problems when different conditions are imposed on different parts of boundary of the domain (e.g., Dirichlet, Neumann, and Newton or other homogeneous or nonhomogeneous conditions).

The general discussion is illustrated by a classic example for the domain Ω in $\mathbb{R}^m$ with Lipschitz boundary $\partial\Omega$. The conjugation problem arising in the diffraction theory and its abstract analog are formulated.

2.1. The general Green formula for mixed boundary-value problems. If the boundary Γ of the domain $\Omega \subset \mathbb{R}^m$ consists of two parts, namely, $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_2 = \Gamma \setminus \overline{\Gamma}_1$, then in the classic case, i.e., for the smooth $\Gamma = \partial\Omega$ and smooth functions η and u, we can rewrite formula (1.2) in the following form:

$$(\eta, u - \Delta u)_{L_2(\Omega)} = (\eta, u)_{H^1(\Omega)} - \sum_{k=1}^{2} \left(\gamma_k\eta, \partial_k u\right)_{L_2(\Gamma_k)},$$

$$\gamma_k\eta := \eta\big|_{\Gamma_k}, \quad \partial_k u := \frac{\partial u}{\partial n}\Big|_{\Gamma_k}, \quad k = 1, 2.$$

It is useful for studying of mixed boundary-value problems where, e.g., the Dirichlet boundary condition is imposed on one part $\Gamma = \partial\Omega$ of the boundary and another boundary condition (e.g., Neumann or Newton conditions) is imposed on the other part.

Now we derive a similar Green formula in an abstract form based on formula (1.6) and the constructions from [46, p. 191-192] (see also [27]).

Let the conditions of Theorem 1.1 hold. Let p_1 be a continuous operator at G_+ and $p_2 := I - p_1$. Then the operators p_k, $k = 1, 2$, are continuous from G_+ to $(\widetilde{G_+})_k := p_k G_+$. Let us introduce the operators

$$\widetilde{\gamma}_k := p_k\gamma, \quad \widetilde{\partial}_k := p_k^*\partial, \text{ and } \quad p_k^* := (\widetilde{G}_+)_k^* \to (G_+)^*, \quad k = 1, 2. \tag{2.1}$$

Lemma 2.1. *The Green formula holds with the above assumptions*:

$$\langle \eta, Lu\rangle_E = (\eta, u)_F - \sum_{k=1}^{2} \langle \widetilde{\gamma}_k \eta, \widetilde{\partial}_k u\rangle_G \quad \forall \eta, u \in F. \tag{2.2}$$

Proof. The proof is quite simple. By construction,

$$\gamma = (p_1 + p_2)\gamma = \widetilde{\gamma}_1 + \widetilde{\gamma}_2$$

and it follows that

$$\begin{aligned}\langle \gamma\eta, \partial u\rangle_G &= \langle (\widetilde{\gamma}_1 + \widetilde{\gamma}_2)\eta, \partial u\rangle_G = \langle \widetilde{\gamma}_1\eta, \partial u\rangle_G + \langle \widetilde{\gamma}_2\eta, \partial u\rangle_G \\ &= \sum_{k=1}^{2} \langle p_k\gamma\eta, \partial u\rangle_G = \sum_{k=1}^{2} \langle p_k^2\gamma\eta, \partial u\rangle_G = \sum_{k=1}^{2} \langle p_k\gamma\eta, p_k^*\partial u\rangle_G = \sum_{k=1}^{2} \langle \widetilde{\gamma}_k\eta, \widetilde{\partial}_k u\rangle_G. \end{aligned} \tag{2.3}$$

□

Remark 2.1. It follows from the proof of Lemma 2.1 that if there are several complementary projectors, for example, their amount is equal to q, i.e.,

$$p_k = p_k^2 : \; G_+ \to (\widetilde{G}_+)_k := p_k G_+, \; k = 1, \ldots, q, \quad \sum_{k=1}^{q} p_k = I, \tag{2.4}$$

then at the right-hand part of (2.2), the summation with respect to k is from $k = 1$ to $k = q$.

Introduce the notation $\overline{1, q} := 1, \ldots, q$.

In problems of mathematical physics (see Sec. 2.2), the operators p_k can have the following structure:

$$p_k = \omega_k \rho_k, \quad k = \overline{1, q}, \tag{2.5}$$

where

$$\rho_k : \; G_+ \to (G_+)_k \tag{2.6}$$

is the operator of reducing the space $(G_+)_k = \rho_k G_+$ (the operators of reducing to a part of the boundary of the domain). Here

$$G = \bigoplus_{k=1}^{q} G_k, \quad (G_+)_k \hookrightarrow G_k \tag{2.7}$$

and $\omega_k : \; (G_+)_k \to (\widetilde{G}_+)_k$ is the operator of "reducing by zero" from $(G_+)_k$ to the subspace $(\widetilde{G}_+)_k \subset G_+$ (from a part of the boundary to the whole boundary), i.e.,

$$\omega_k (G_+)_k = \omega_k \rho_k G_+ = p_k G_+ = (\widetilde{G}_+)_k. \tag{2.8}$$

Moreover, it is assumed in (2.5) that ω_k is the right inverse operator for ρ_k, i.e.,

$$\rho_k \omega_k = I_k \quad (\text{in } (G_+)_k), \; k = \overline{1, q}, \tag{2.9}$$

and the operators ρ_k and ω_k are continuous from G_+ onto $(G_+)_k$ and from $(G_+)_k$ onto $(\widetilde{G}_+)_k$, respectively.

Lemma 2.2. *If the formulated above conditions hold, then the Green formula* (2.2) (*taking into account Remark* 2.1) *has the following form*:

$$\langle \eta, Lu \rangle_E = (\eta, u)_F - \sum_{k=1}^{q} \langle \gamma_k \eta, \partial_k u \rangle_{G_k} \quad \forall \eta, u \in F, \tag{2.10}$$

$$\gamma_k \eta := \rho_k \gamma \eta, \quad \partial_k u := \omega_k^* \partial u, \tag{2.11}$$

where γ_k is an abstract trace operator to a part of the boundary of the domain and ∂_k is an abstract operator of a outward normal derivative that acts on a part of the boundary of the domain.

Proof. Converting the expression under the summation sign in (2.2) according to conditions (2.5)–(2.9) and taking into account (2.3), we have

$$\langle \widetilde{\gamma}_k \eta, \widetilde{\partial}_k u \rangle_G = \langle p_k \gamma \eta, \partial u \rangle_G = \langle \omega_k \rho_k \gamma \eta, \partial u \rangle_G.$$

We see that the right-hand side is a bounded linear functional with respect of $\rho_k \gamma \eta = \gamma_k \eta \in (G_+)_k$, as ω_k is continuous and

$$|\langle \omega_k \rho_k \gamma \eta, \partial u \rangle_G| \leq \|\omega_k\| \cdot \|\rho_k \gamma \eta\|_{(G_+)_k} \cdot \|\partial u\|_{(G_+)^*}.$$

Therefore, this functional in the "inner product G_k" has the following form:

$$\langle \omega_k \rho_k \gamma \eta, \partial u \rangle_G = \langle \rho_k \gamma \eta, \omega_k^* \partial u \rangle_{G_k} =: \langle \gamma_k \eta, \partial_k u \rangle_{G_k}.$$

We use this notation for ∂_k because in the smooth classic case (for the boundary of the domain and the function u), the outward normal derivative arises at a part of the boundary. □

2.2. A classic example. Let us prove the statements of Lemmas 2.1 and 2.2 for the triple of spaces (1.7), i.e., for the elements of $H^1(\Omega), \Omega \subset \mathbb{R}^m$, at the domain Ω with a Lipschitz boundary.

Let us recall that a bounded domain $\Omega \subset \mathbb{R}^m$ has a Lipschitz boundary $\Gamma = \partial\Omega$ if there is a neighborhood of every boundary point and an orthogonal coordinate system for this neighborhood $0y_1 \ldots y_m$ such that the equation of the part of the boundary $\partial\Omega$ in this neighborhood has the following form: $y_m = f(y_1, \ldots, y_{m-1})$, where f is a Lipschitz function

$$|f(y_1, \ldots, y_{m-1}) - f(z_1, \ldots, z_{m-1})| \leq C \left(\sum_{k=1}^{m-1} |y_k - z_k|^2 \right)^{1/2}.$$

Let us introduce a functional space $H^1(\Omega)$ with a standard norm in the domain Ω:

$$\|u\|^2_{H^1(\Omega)} := \int_\Omega \left(|\nabla u|^2 + |u|^2 \right) d\Omega, \quad |\nabla u|^2 = \sum_{k=1}^{m} \left| \frac{\partial u}{\partial x_k} \right|^2$$

and its subspace, which is the kernel of the operator γ:

$$H^1_0(\Omega) := \ker \gamma = \left\{ u \in H^1(\Omega) : \ \gamma u = u|_\Gamma = 0 \right\}. \tag{2.12}$$

As is known, the set

$$H^1_h(\Omega) := \left\{ u \in H^1(\Omega) : \ u - \Delta u = 0 \text{ (in } \Omega) \right\} \tag{2.13}$$

is an orthogonal complement to $H^1_0(\Omega)$ at $H^1(\Omega)$. This set is called a subspace of harmonic functions. Thus, we have an orthogonal decomposition

$$H^1(\Omega) = H^1_0(\Omega) \oplus H^1_h(\Omega). \tag{2.14}$$

The next statement (see [18]) is very important.

Theorem 2.1 (E. Gagliardo)**.** *Let a bounded domain $\Omega \subset \mathbb{R}^m$ have a Lipschitz boundary $\Gamma = \partial\Omega$. Introduce the Hilbert space $H^{1/2}(\Gamma)$ with the following squared norm at this boundary:*

$$\|\varphi\|^2_{H^{1/2}(\Gamma)} := \int\limits_{\Gamma} |\varphi|^2 \, d\Gamma + \int\limits_{\Gamma_x} d\Gamma_x \int\limits_{\Gamma_y} d\Gamma_y \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{m+1}}. \tag{2.15}$$

Then the operator γ defined as

$$\gamma u := u\big|_{\Gamma} \quad \forall u \in H^1(\Omega)$$

is bounded from $H^1(\Omega)$ in $H^{1/2}(\Gamma)$, i.e., the following estimate holds:

$$\|\gamma u\|_{H^{1/2}(\Gamma)} \le c_1 \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega). \tag{2.16}$$

Conversely, for every function $\varphi \in H^{1/2}(\Gamma)$, there is a function $u \in H^1(\Omega)$ (which is ambiguously defined by φ) such that

$$\gamma u = \varphi, \quad \|u\|_{H^1(\Omega)} \le c_2 \|\varphi\|_{H^{1/2}(\Gamma)}. \tag{2.17}$$

In this case, the space $H^{1/2}(\Gamma)$ is embedded compactly in $L_2(\Gamma)$.

To derive the Green formula for mixed boundary-value problems, we select simple open parts with Lipschitz boundaries $\partial\Gamma_k$ at the surface $\Gamma = \partial\Omega$ of the domain Ω. It is sufficient for the applications that $\partial\Gamma_k$ are piecewise smooth with nonzero interior and exterior angles. We have

$$\Gamma = \left(\bigcup_{k=1}^{q} \Gamma_k \right) \cup \left(\bigcup_{k=1}^{q} \partial\Gamma_k \right), \quad \operatorname{mes} \Gamma_k > 0, \ \operatorname{mes} \partial\Gamma_k = 0, \ k = \overline{1,q}. \tag{2.18}$$

Let us introduce the operator ρ_k of restriction from Γ to Γ_k:

$$\rho_k \varphi := \varphi\big|_{\Gamma_k} \quad \forall \varphi \in H^{1/2}(\Gamma). \tag{2.19}$$

This operator maps every function $\varphi \in H^{1/2}(\Gamma)$ to its part φ_k, which is given on $\Gamma_k \subset \Gamma$.

Lemma 2.3. *The operator of restriction*

$$\rho_k : \ H^{1/2}(\Gamma) \to H^{1/2}(\Gamma_k), \quad k = \overline{1,q}, \tag{2.20}$$

is bounded and has the following norm:

$$\|\rho_k\|_{H^{1/2}(\Gamma) \to H^{1/2}(\Gamma_k)} \le 1. \tag{2.21}$$

Proof. Statements (2.20) and (2.21) follow from definition (2.15) of the space norm $H^{1/2}(\Gamma)$ since

$$\|\varphi\|^2_{H^{1/2}(\Gamma_k)} \le \|\varphi\|^2_{H^{1/2}(\Gamma)}, \quad \Gamma_k \subset \Gamma \quad \forall \varphi \in H^{1/2}(\Gamma).$$

□

Let us introduce the subspaces

$$H^1_{0,\Gamma \setminus \Gamma_k}(\Omega) := \left\{ u \in H^1(\Omega) : \ u = 0 \text{ on } \Gamma \setminus \Gamma_k \right\}, \quad k = \overline{1,q}. \tag{2.22}$$

Since

$$H^1_{0,\Gamma \setminus \Gamma_k}(\Omega) \supset H^1_0(\Omega) \quad \forall k = \overline{1,q},$$

we have that $H^1_{0,\Gamma \setminus \Gamma_k}(\Omega)$ is dense at $L_2(\Omega)$ for every $k = \overline{1,q}$.

It is natural for problems of mathematical physics for the elements $u \in H^1(\Omega)$ to accept that

$$\frac{\partial u}{\partial n}\Big|_{\Gamma_k} \in H^{-1/2}(\Gamma_k), \quad k = \overline{1,q}. \tag{2.23}$$

The collection of such sets of derivatives at the parts Γ_k of the boundary $\Gamma = \partial\Omega$ is wider than the collection of normal derivatives $\frac{\partial u}{\partial n}\Big|_{\Gamma}$, $u \in H^1(\Omega)$ as in (2.23) these finite-ordered generalized functions defined at Γ_k cannot present a generalized function $\left.\frac{\partial u}{\partial n}\right|_\Gamma \in H^{-1/2}(\Gamma)$ which is composed from them. Therefore, the collection of trial functions given at Γ can be narrower than $H^{1/2}(\Gamma)$. Consequently, the Green formula (2.10) is valid for some subset of the space $H^1(\Omega)$ (see Theorem 1.2) in mixed boundary-value problems with conditions (2.23).

Note that we take the space $H^{-1/2}(\Gamma_k)$ of elements with the norm

$$\|\psi\|_{H^{-1/2}(\Gamma_k)} := \inf_{\widehat{\psi}|_{\Gamma_k}=\psi} \|\widehat{\psi}\|_{H^{-1/2}(\Gamma)} \tag{2.24}$$

in (2.23) as $H^{-1/2}(\Gamma_k)$ (see, e.g., [2, formula 5 and the end of Sec. 4]). The next statement (see [2, 49] and [47, Theorem 2.1]) is valid for elements from $H^{-1/2}(\Gamma_k)$ with Lipschitz boundary $\partial\Gamma_k$: there is a linear operator $\mathcal{E}$ of extension of functions from $H^{-1/2}(\Gamma_k)$ from Γ_k onto the whole Γ by functions from $H^{-1/2}(\Gamma)$ and

$$\|\mathcal{E}\psi\|_{H^{-1/2}(\Gamma)} \le c\|\psi\|_{H^{-1/2}(\Gamma_k)} \quad \forall\psi \in H^{-1/2}(\Gamma_k). \tag{2.25}$$

Note that this operator $\mathcal{E}$ was introduced by Rychkov (see [2, 49]), and it has a unique property: it is bounded from the space $H_p^s(\Omega)$ to the space $H_p^s(\mathbb{R}^m)$ and from $H_p^s(\Gamma_k)$ to $H_p^s(\Gamma)$ and it does not depend on the indices s and p, $|s| \le 1$, $1 < p < \infty$.

Using these facts, let us consider the sesquilinear form

$$[\varphi, \psi]_\Gamma := \langle \varphi, \mathcal{E}\psi\rangle_{L_2(\Gamma)} \quad \forall\varphi \in H^{1/2}(\Gamma)\ \forall\psi \in H^{-1/2}(\Gamma_k). \tag{2.26}$$

Here $\mathcal{E}\psi \in H^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))^*$ (the conjugation is with respect to the form $L_2(\Gamma)$). The following estimate is valid for this form by virtue of (2.25):

$$|[\varphi, \psi]_\Gamma| \le \|\varphi\|_{H^{1/2}(\Gamma)} \cdot \|\mathcal{E}\| \cdot \|\psi\|_{H^{-1/2}(\Gamma_k)}. \tag{2.27}$$

Now let $\varphi = \gamma\eta$ and $\eta \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)$. Then $\gamma\eta \in H^{1/2}(\Gamma)$ and $\gamma\eta = 0$ at $\Gamma \setminus \Gamma_k$. Let us consider a sequence $\{\psi_j\}_{j=1}^\infty$ of elements from $L_2(\Gamma_k)$ converging to the element $\psi \in H^{-1/2}(\Gamma_k)$ by the norm $H^{-1/2}(\Gamma_k)$ (such a sequence exists since $L_2(\Gamma_k)$ is dense at $H^{-1/2}(\Gamma_k)$). By virtue of (2.26) and the fact that $\mathcal{E}:\ L_2(\Gamma_k) \to L_2(\Gamma)$ is a bounded operator, we have for a fixed φ:

$$\langle \varphi, \mathcal{E}\psi\rangle_{L_2(\Gamma)} = \lim_{j\to\infty} \langle \varphi, \mathcal{E}\psi_j\rangle_{L_2(\Gamma)} = \lim_{j\to\infty} (\varphi, \mathcal{E}\psi_j)_{L_2(\Gamma)} = \lim_{j\to\infty} (\rho_k\varphi, \psi_j)_{L_2(\Gamma_k)} =: \langle \rho_k\varphi, \psi\rangle_{L_2(\Gamma_k)}, \tag{2.28}$$

where $\rho_k:\ H^{1/2}(\Gamma) \to H^{1/2}(\Gamma_k)$ is an operator of restriction from Lemma 2.3, and on the right-hand side, we have an extension of an inner product at $L_2(\Gamma_k)$ to elements $\rho_k\varphi = \rho_k\gamma\eta =: \gamma_k\eta$, $\eta \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)$ and $\psi \in H^{-1/2}(\Gamma_k)$ (with respect to $L_2(\Gamma_k)$).

Note that for a fixed φ, the right-hand side in (2.28) does not depend on the type of extension of the element $\psi \in H^{-1/2}(\Gamma_k)$ to the element $\widehat{\psi} \in H^{-1/2}(\Gamma)$, since the values $(\rho_k\varphi, \psi_j)_{L_2(\Gamma_k)}$ are determined by the elements ψ_j, given at Γ_k. Therefore, the limit expression depends only on ψ, instead of $\widehat{\psi}$.

Using these facts, let us consider the following auxiliary mixed boundary-value problem:

$$u - \Delta u = 0 \ (\text{in } \Omega), \quad u = 0 \ (\text{on } \Gamma \setminus \Gamma_k), \quad \left.\frac{\partial u}{\partial n}\right|_{\Gamma_k} = \psi_k \ (\text{on } \Gamma_k). \tag{2.29}$$

A function $u \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)$ for which the following equality is valid for every $\eta \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)$:

$$(\eta, u)_{H^1(\Omega)} = (\gamma_k\eta, \psi_k)_{L_2(\Gamma_k)}, \quad \gamma_k := \rho_k\gamma,\ \psi_k \in L_2(\Gamma_k) \tag{2.30}$$

is called a general solution of problem (2.29).

A function $u \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)$ for which the equality

$$(\eta, u)_{H^1(\Omega)} = \langle \gamma_k\eta, \psi_k\rangle_{L_2(\Gamma_k)}, \quad \psi_k \in H^{-1/2}(\Gamma_k) \tag{2.31}$$

is valid for every $\eta \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)$ is called a weak solution of problem (2.29). The right-hand side here is the value of the functional $\psi_k \in H^{-1/2}(\Gamma_k)$ on the element $\gamma_k\eta = \rho_k\gamma\eta$, $\eta \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)$. (It is obvious that the classic and general solutions of problem (2.29) are weak in the sense of definition (2.31).)

Lemma 2.4. *For every $\psi_k \in H^{-1/2}(\Gamma_k)$, there is a unique weak solution of problem* (2.29) *and*

$$u \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega) \cap H^1_h(\Omega) =: M_k(\Omega) \subset H^1_h(\Omega). \tag{2.32}$$

Proof. The proof is based on the fact that for every $\eta \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)$, the right-hand side in (2.31) is a bounded linear functional $H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)$. In fact, taking into account relations (2.28), (2.26), and (2.27), and Theorem 2.1 (see inequalities (2.16)), we have

$$|\langle\gamma_k\eta, \psi_k\rangle_{L_2(\Gamma_k)}| = |\langle\rho_k\gamma\eta, \psi_k\rangle_{L_2(\Gamma_k)}| = |\langle\gamma\eta, \mathcal{E}\psi\rangle_{L_2(\Gamma)}| \le$$
$$\le \|\gamma\eta\|_{H^{1/2}(\Gamma)} \cdot \|\mathcal{E}\| \cdot \|\psi\|_{H^{-1/2}(\Gamma_k)} \le \Big(c_1\|\mathcal{E}\| \cdot \|\psi\|_{H^{-1/2}(\Gamma_k)}\Big)\, \|\eta\|_{H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)}.$$

Therefore, there exists a unique element $u =: T_k\psi_k \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)$ such that Eq. (2.31) is valid. If we set in (2.31) $\eta \in H^1_0(\Omega)$, then we have $(\eta, u)_{H^1(\Omega)} = 0$ and, by virtue of (2.14), we see $u \in H^1_h(\Omega)$. Property (2.32) follows from here. □

Lemma 2.4 implies that the operator T_k acts from $H^{-1/2}(\Gamma_k)$ to $M_k(\Omega)$. If we set in (2.31) $\eta \in M_k(\Omega)$, then we have the equality

$$(\eta, T_k\psi_k)_{H^1(\Omega)} = \langle\gamma^0_k\eta, \psi_k\rangle_{L_2(\Gamma_k)} \quad \forall\eta \in M_k(\Omega),\ \psi_k \in H^{-1/2}(\Gamma_k), \tag{2.33}$$

where

$$\gamma^0_k := \gamma_k\big|_{M_k(\Omega)} = (\rho_k\gamma)\big|_{M_k(\Omega)}. \tag{2.34}$$

It is obvious from the construction that elements of the type $\gamma^0_k u$ for all $u \in M_k(\Omega)$ have the following properties. First, they belong to the space $H^{1/2}(\Gamma_k)$ (see Theorem 2.1 and Lemma 2.3). Second, being extended by zero from Γ_k to the whole Γ, they belong to $H^{1/2}(\Gamma)$. Moreover, it is obvious that there is a one-to-one correspondence between the elements u from $M_k(\Omega)$ and the collection of elements of the type $\gamma^0_k u$. Indeed, if $u \in M_k(\Omega)$ and $\gamma^0_k u = 0$, then $\gamma u = 0$ on $\partial\Omega$ and hence $u = 0$. (On the other hand, if $u \equiv 0$, then $\gamma^0_k u = 0$.)

Let us denote by $\widetilde{H}^{1/2}(\Gamma_k)$ the collection of elements of the form

$$\widetilde{H}^{1/2}(\Gamma_k) := \{\gamma^0_k u :\ u \in M_k(\Omega)\} \subset H^{1/2}(\Gamma_k). \tag{2.35}$$

Lemma 2.5. *The set $\widetilde{H}^{1/2}(\Gamma_k)$ is dense in $L_2(\Gamma_k)$.*

Proof. Assume that there exists an element $\varphi_0 \in L_2(\Gamma_k)$ that is orthogonal to all the elements from $\widetilde{H}^{1/2}(\Gamma_k)$, i.e.,

$$(\gamma^0_k\eta, \varphi_0)_{L_2(\Gamma_k)} = 0 \quad \forall\eta \in M_k(\Omega).$$

Then, by virtue of (2.12), this property is valid for any $\eta \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega)$. Therefore, from (2.31), taking into consideration the equality

$$\langle\gamma^0_k\eta, \varphi_0\rangle_{L_2(\Gamma_k)} = (\gamma^0_k\eta, \varphi_0)_{L_2(\Gamma_k)},$$

we have

$$(\eta, T_k\varphi_0)_{H^1(\Omega)} = 0 \quad \forall\eta \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega).$$

Consequently, $T_k\varphi_0 = 0$, and $\varphi_0 = 0$. □

Let us consider the operator

$$C_k := \gamma_k^0 T_k : \ H^{-1/2}(\Gamma_k) \to \widetilde{H}^{1/2}(\Gamma_k). \tag{2.36}$$

According to the construction, there is a one-to-one correspondence between $\mathcal{D}(C_k) = H^{-1/2}(\Gamma_k)$ and its value area $\mathcal{R}(C_k) = \widetilde{H}^{1/2}(\Gamma_k)$. Taking into account this fact and the fact that there is an isomorphism between $M_k(\Omega)$ and $\widetilde{H}^{1/2}(\Gamma_k)$, we introduce the Hilbert-space structure on $\widetilde{H}^{1/2}(\Gamma_k)$. Here

$$(\alpha, \beta)_{\widetilde{H}^{1/2}(\Gamma_k)} := (\eta, u)_{H^1(\Omega)}, \quad \eta, u \in M_k(\Omega), \ \gamma_k^0 \eta = \alpha, \ \gamma_k^0 u = \beta. \tag{2.37}$$

Then from (2.33) it follows that as $\eta \in T_k \widetilde{\psi}_k, \ \widetilde{\psi}_k \in H^{-1/2}(\Gamma_k)$, the following equality holds:

$$(T_k \widetilde{\psi}_k, T_k \psi_k)_{H^1(\Omega)} = \langle C_k \widetilde{\psi}_k, \psi_k \rangle_{L_2(\Gamma_k)}, \quad \widetilde{\psi}_k, \ \psi_k \in H^{-1/2}(\Gamma_k). \tag{2.38}$$

Taking into account definition (2.37), we can rewrite this relation in the following form:

$$(\alpha, \beta)_{\widetilde{H}^{1/2}(\Gamma_k)} = (\eta, u)_{H^1(\Omega)} = \langle \alpha, C_k^{-1} \beta \rangle_{L_2(\Gamma_k)} \quad \forall \alpha, \beta \in \widetilde{H}^{1/2}(\Gamma_k), \tag{2.39}$$

$$\eta = T_k \widetilde{\psi}_k, \ \gamma_k^0 \eta = \alpha, \ u = T_k \psi_k, \ \gamma_k^0 T_k \psi_k = C_k \psi_k = \beta, \ \psi_k = C_k^{-1} \beta, \quad \widetilde{\psi}_k, \ \psi_k \in H^{-1/2}(\Gamma_k). \tag{2.40}$$

Before we formulate the following statements, let us recall the notion of a Hilbert pair of spaces and corresponding properties (see, e.g., [31, pp. 32–47] and [37, p. 251]).

We say that Hilbert spaces F and E (with the inner products $(\cdot,\cdot)_F$ and $(\cdot,\cdot)_E$ respectively) form a Hilbert pair $(F; E)$ if condition (1.4) is valid, i.e., F is continuous and densely embedded in E and there exists an $a > 0$ such that

$$\|u\|_E \le a \|u\|_F \quad \forall u \in F.$$

Using the pair $(F; E)$, we can form a noncontinuous self-adjoint operator A (an operator of a Hilbert pair) such that $F = \mathcal{D}(A^{1/2}), \ \mathcal{D}(A) \subset F$. It can be determine by the following equality:

$$(u, v)_F = (u, Av)_E, \quad u \in F, \ v \in \mathcal{D}(A) \subset F.$$

Using the operator A, we can form a Hilbert spaces scale $E^\alpha, \ \alpha \in \mathbb{R}$, such that $E^0 = E, \ E^{1/2} = F, \ E^{-1/2} = F^*$ (with respect to the form of E), and the operator A is considered to act in this scale. In this case,

$$AE^{1/2} = AF = E^{-1/2} = F^*, \quad A^{1/2} F = E^0 = E, \quad A^{1/2} E^0 = F^*;$$

Moreover, the triple of spaces

$$F \hookrightarrow E \hookrightarrow F^*$$

forms an equipment of the Hilbert space E. Then the operator A of the Hilbert pair $(F; E)$, given on $\mathcal{D}(A) = F$, can be determined from the following equality:

$$(u, v)_F = (A^{1/2} u, A^{1/2} v)_E = \langle u, Av \rangle_E \quad \forall u, v \in F = E^{1/2}, \tag{2.41}$$

where $\langle u, Av \rangle_E$ is the value of the bounded linear functional $Av \in F^*$ on the element $u \in F$.

Lemma 2.6. *The operator $C_k^{-1} = (\gamma_k^0 T_k)^{-1}$ with $\mathcal{D}(C_k^{-1}) = \widetilde{H}^{1/2}(\Gamma_k)$ and $\mathcal{R}(C_k^{-1}) = \mathcal{D}(C_k) = H^{-1/2}(\Gamma_k)$ is the operator of a Hilbert pair $(\widetilde{H}^{1/2}(\Gamma_k); L_2(\Gamma_k))$.*

Proof. Recall that $\widetilde{H}^{1/2}(\Gamma_k)$ is dense in $L_2(\Gamma_k)$ (Lemma 2.5) and, by construction, it is a complete space with respect to the norm induced by the inner product (2.37). Since $\widetilde{H}^{1/2}(\Gamma_k) \subset H^{1/2}(\Gamma_k)$, by virtue of Lemma 2.3 and relations (2.15), (2.16), and (2.37), we have

$$\|\varphi\|_{L_2(\Gamma_k)} \le \|\varphi\|_{H^{1/2}(\Gamma_k)} \le \|\widehat{\varphi}\|_{H^{1/2}(\Gamma)} \le c_1 \|u\|_{M_k(\Omega)} = c_1 \|\varphi\|_{\widetilde{H}^{1/2}(\Gamma_k)}, \tag{2.42}$$

$$\varphi = \gamma_k^0 u, \ u \in M_k(\Omega), \ \varphi \in \widetilde{H}^{1/2}(\Gamma_k),$$

where $\widehat{\varphi}$ is the function $\varphi \in \widetilde{H}^{1/2}(\Gamma_k)$ extended by zero on $\Gamma \setminus \Gamma_k$ and now defined on the whole Γ.

Hence, $\widetilde{H}^{1/2}(\Gamma_k)$ and $L_2(\Gamma_k)$ form a Hilbert pair of spaces $(\widetilde{H}^{1/2}(\Gamma_k); L_2(\Gamma_k))$. By definition (2.41), for the operator A of this pair, the following equality should holds:

$$(\alpha, \beta)_{\widetilde{H}^{1/2}(\Gamma_k)} = \langle \alpha, A\beta \rangle_{L_2(\Gamma_k)} \quad \forall \alpha, \beta \in \widetilde{H}^{1/2}(\Gamma_k).$$

Comparing this equality with (2.39), we conclude that $A = C_k^{-1} : \widetilde{H}^{1/2}(\Gamma_k) \to H^{-1/2}(\Gamma_k)$ is the operator of the Hilbert pair $(\widetilde{H}^{1/2}(\Gamma_k); L_2(\Gamma_k))$. □

It follows from this lemma that the triple of spaces

$$\widetilde{H}^{1/2}(\Gamma_k) \subset L_2(\Gamma_k) \subset H^{-1/2}(\Gamma_k) \tag{2.43}$$

forms an equipment of the space $L_2(\Gamma_k)$.

Note that the norm in $\widetilde{H}^{1/2}(\Gamma_k)$ introduced by formula (2.37) is "stronger" than the standard norm in $H^{1/2}(\Gamma_k)$. Indeed, this fact follows from (2.42):

$$\|\varphi\|_{H^{1/2}(\Gamma_k)} \le c_1 \|\varphi\|_{\widetilde{H}^{1/2}(\Gamma_k)} \quad \forall \varphi \in \widetilde{H}^{1/2}(\Gamma_k). \tag{2.44}$$

Introduce the operator ω_k of extension by zero on $\Gamma \backslash \Gamma_k$ on elements from $\widetilde{H}^{1/2}(\Gamma_k)$ acting according to the following rule:

$$\omega_k \varphi_k := \begin{cases} \varphi_k & \text{on } \Gamma_k, \\ 0, & \text{on } \Gamma \setminus \Gamma_k \ \forall \varphi_k \in \widetilde{H}^{1/2}(\Gamma_k). \end{cases} \tag{2.45}$$

Lemma 2.7. *An operator of extension by zero from Γ_k to Γ, which is considered on $\mathcal{D}(\omega_k) := \widetilde{H}^{1/2}(\Gamma_k)$, is a continuous operator acting from $\widetilde{H}^{1/2}(\Gamma_k)$ onto $H^{1/2}(\Gamma)$, so that*

$$\|\omega_k \varphi_k\|_{H^{1/2}(\Gamma)} \le c_1 \|\varphi_k\|_{\widetilde{H}^{1/2}(\Gamma_k)}, \tag{2.46}$$

where c_1 is the constant from inequality (2.16).

Proof. This statement has already been proved in the proof of Lemma 2.6. Indeed, by virtue of (2.45), we have in (2.42) that $\widehat{\varphi} = \omega_k \varphi_k$. Formula (2.46) follows from this. □

Remark 2.2. As is known (see, e.g., [41, p. 78] and [57, pp. 116]), even for the smooth Γ the operator of extension by zero from some part of Γ_k to the whole Γ is not continuous from $H^{1/2}(\Gamma_k)$ to $H^{1/2}(\Gamma)$. Nevertheless, this operator is bounded for a given problem on the solutions of auxiliary problem (2.29), i.e., on the elements $\gamma_k^0 u,\ u \in M_k(\Omega)$.

Let us introduce the following classes of functions:

$$\widetilde{H}^1(\Omega) := H_0^1(\Omega) \oplus \left((\dot{+})_{k=1}^m M_k(\Omega)\right), \quad M_k(\Omega) = H_h^1(\Omega) \cap H^1_{0,\Gamma\backslash\Gamma_k}(\Omega), \tag{2.47}$$

$$\widetilde{H}^{1/2}(\Gamma) := \left\{\varphi \in H^{1/2}(\Gamma) :\ \rho_k \varphi \in \widetilde{H}^{1/2}(\Gamma_k),\ k = \overline{1,q}\right\}. \tag{2.48}$$

Definition 2.1. A trace γu of the element $u \in H^1(\Omega)$ is called *regular* with respect to a decomposition $\Gamma = \partial\Omega$ into parts $\Gamma_k,\ k = \overline{1,q}$ (see (2.18)), if for any $k = \overline{1,q}$ the element $\gamma_k u = \rho_k \gamma u \in \widetilde{H}^{1/2}(\Gamma_k)$, i.e., it can be extended by zero to the whole Γ in the class $H^{1/2}(\Gamma)$.

According the constructions and definition (2.47) and (2.48), we see that the elements from $\widetilde{H}^1(\Omega)$ have a regular trace, i.e., for any $u \in \widetilde{H}^1(\Omega)$ we have

$$u = u_0 + u_1 + \ldots u_q, \quad u_0 \in H_0^1(\Omega),\ u_k \in M_k(\Omega),\ k = \overline{1,q}, \tag{2.49}$$

$$\gamma u_0 = 0, \quad \gamma_k u_k = \gamma_k^0 u_k = \varphi_k \in \widetilde{H}^{1/2}(\Gamma_k), \quad \gamma_k u_j = 0\ (k \neq j), \quad j,k = \overline{1,q}.$$

In this case, the elements $\gamma u \in \widetilde{H}^{1/2}(\Gamma)$ have restrictions at Γ_k and can be extended by zero to the whole Γ in the class $H^{1/2}(\Gamma)$.

Let us consider the following triple of spaces and the trace operator

$$E = L_2(\Omega),\ F = \widetilde{H}^1(\Omega),\ G = L_2(\Gamma),\ \gamma u := u\big|_\Gamma,\ u \in \widetilde{H}^1(\Omega). \tag{2.50}$$

The following properties are valid.

1°. $\widetilde{H}^1(\Omega)$ is dense in $L_2(\Omega)$ ($\widetilde{H}^1(\Omega) \supset H_0^1(\Omega)$) and

$$\|u\|_{L_2(\Omega)} \le c\|u\|_{H^1(\Omega)} = c\|u\|_{\widetilde{H}^1(\Omega)} \quad \forall u \in \widetilde{H}^1(\Omega).$$

2°. The operator $\gamma:\ \widetilde{H}^1(\Omega) \to \widetilde{H}^{1/2}(\Gamma)$ is bounded, $\widetilde{H}^{1/2}(\Gamma)$ is dense in $L_2(\Gamma)$, and (according to the Sobolev trace theorem)

$$\|\gamma u\|_{L_2(\Gamma)} \le \widetilde{c}\|u\|_{H^1(\Omega)} \quad \forall u \in \widetilde{H}^1(\Omega).$$

3°. By virtue of Lemmas 2.3 and 2.7, for every $k = \overline{1,q}$, the operator $p_k = \omega_k \rho_k$ is bounded in the space $G_+ := \widetilde{H}^{1/2}(\Gamma)$, $p_k^2 = p_k$ and $\rho_k \omega_k$ is the identity operator in $\widetilde{H}^{1/2}(\Gamma_k)$ by construction.

According to Lemma 2.2 we can conclude the following.

Theorem 2.2. *The following Green formula takes place for the triple of spaces* $L_2(\Omega),\ \widetilde{H}^1(\Omega),\ L_2(\Gamma)$ *(where* $\Gamma = \partial\Omega$*) and the trace operator* $\gamma:\ \widetilde{H}^1(\Omega) \to L_2(\Gamma)$ $(\gamma\eta := \eta\big|_\Gamma,\ \eta \in \widetilde{H}^1(\Omega))$ *in the domain* $\Omega \subset \mathbb{R}^m$ *with a Lipschitz boundary* Γ:

$$\langle \eta, u - \Delta u\rangle_{L_2(\Omega)} = (\eta, u)_{H^1(\Omega)} - \sum_{k=1}^{q} \langle \gamma_k \eta, \partial_k u\rangle_{L_2(\Gamma_k)} \quad \forall \eta, u \in \widetilde{H}^1(\Omega), \tag{2.51}$$

$$\Gamma = \left(\bigcup_{k=1}^{q} \Gamma_k\right) \cup \left(\bigcup_{k=1}^{q} \partial\Gamma_k\right), \qquad \mathrm{mes}(\Gamma_k \cap \Gamma_j) = 0\ (k \ne j), \qquad \Delta u \in (H^1(\Omega))^*,$$

$$\gamma_k \eta := \eta\big|_{\Gamma_k} \in \widetilde{H}^{1/2}(\Gamma_k), \qquad \partial_k u := \frac{\partial u}{\partial n}\bigg|_{\Gamma_k} \in H^{-1/2}(\Gamma_k), \qquad k = \overline{1,q}.$$

Note that the Green formula in the form (2.51) for mixed boundary-value problems can be proved with help of constructions from [31, pp. 46] and [30] and Eqs. (2.47)–(2.49) instead of the general approach of Sec. 2.1.

2.3. Multicomponent conjugation problems. We start with the case of conjugation problems for the Helmholtz equation (see [5, 52, 56]).

Consider q bounded domains $\Omega_j,\ j = \overline{1,q}$, with Lipschitz boundaries $\Gamma_j = \partial\Omega_j$ in $\mathbb{R}^m$ $(m \ge 2)$. These domains are adjoint by some parts of their boundaries. Moreover, some parts of boundaries can be free, i.e., they do not border neighboring domains.

Let us denote by $\Gamma_{jj},\ j = \overline{1,q}$ free (external) parts of boundaries Γ_j. We denote by Γ_{jk} the part of the boundary Γ_j that adjoins a part of the boundary of the domain Ω_k $(k \ne j)$. We see that $\Gamma_{jk} = \Gamma_{kj}$. We have the matrix of boundaries

$$(\Gamma_{jk})_{j,k=1}^q\,.$$

We consider its elements as an $(m-1)$-dimensional manifold with a border.

Let us formulate the statement of a multicomponent conjugation problem for the given collection of domains Ω_j and $j = \overline{1,q}$.

It is required to find functions $u_j(x) \in H^1(\Omega_j)$, $j = \overline{1,q}$, such that the following Helmholtz equations are valid in Ω_j:

$$u_j - \Delta u_j + \lambda a_j u_j = 0 \quad (\text{in } \Omega_j),\ j = \overline{1,q}, \tag{2.52}$$

where $\lambda \in \mathbb{C}$ is a parameter and $a_j \in \mathcal{L}\left(H^1(\Omega_j), (H^1(\Omega_j))^*\right)$ are bounded linear operators that are positive definite, i.e.,

$$\langle u_j, a_j u_j\rangle_{L_2(\Omega_j)} \ge c_j \|u_j\|^2_{L_2(\Omega_j)}, \quad c_j > 0, \quad j = \overline{1,q}. \tag{2.53}$$

Boundary conditions of conjugation problems at adjoining and free boundaries can be classified as follows. Let us divide Γ_{jk} into nonintersecting parts Γ_{jkl}, $l=\overline{1,4}$, and set four types of conditions of conjugation as $k>j$ at the boundary Γ_{jk}.

1°. The conditions of the first conjugation problem with a parameter are the following:

$$\gamma_{jk1}u_j=\gamma_{kj1}u_k,\quad \frac{\partial u_j}{\partial n_{jk1}}+\frac{\partial u_k}{\partial n_{kj1}}+\delta_{jk1}\gamma_{jk1}u_j=\mu\alpha_{jk1}\gamma_{jk1}u_j\quad(\text{on }\Gamma_{jk1}),\tag{2.54}$$

where $\mu\in\mathbb{C}$ is the parameter and the trace operators are denoted by γ_{jkl}, $l=\overline{1,4}$, i.e.,

$$\gamma_{jkl}u_j:=u_j\big|_{\Gamma_{jkl}},\quad j=\overline{1,q},\ k>j,\ l=\overline{1,4},\tag{2.55}$$

and $\partial/\partial n_{jkl}$ is the symbol of an outward normal derivative. The bounded operators acting from $H^{1/2}(\Gamma_{jkl})$ to $H^{-1/2}(\Gamma_{jkl})$ are denoted in (2.54) and further by α_{jkl} and δ_{jkl}, $l=\overline{1,4}$. The operators α_{jkl} are positive definite and the operators δ_{jkl} are nonnegative:

$$\langle\gamma_{jkl}u_j,\alpha_{jkl}\gamma_{jkl}u_j\rangle_{L_2(\Gamma_{jkl})}\geq c_{jkl}\|\gamma_{jkl}u_j\|^2_{L_2(\Gamma_{jkl})},\quad c_{jkl}>0;\tag{2.56}$$

$$\langle\gamma_{jkl}u_j,\delta_{jkl}\gamma_{jkl}u_j\rangle_{L_2(\Gamma_{jkl})}\geq 0.\tag{2.57}$$

2°. The conditions of the first conjugation problem without parameters:

$$\gamma_{jk2}u_j=\gamma_{kj2}u_k,\quad \frac{\partial u_j}{\partial n_{jk2}}+\frac{\partial u_k}{\partial n_{kj2}}+\delta_{jk2}\gamma_{jk2}u_j=0\quad(\text{on }\Gamma_{jk2}).\tag{2.58}$$

3°.

$$\frac{\partial u_j}{\partial n_{jk3}}=-\frac{\partial u_k}{\partial n_{kj3}}=-\delta_{jk3}(\gamma_{jk3}u_j-\gamma_{kj3}u_k)+\mu\alpha_{jk3}(\gamma_{jk3}u_j-\gamma_{kj3}u_k)\quad(\text{on }\Gamma_{jk3}).\tag{2.59}$$

4°. The conditions of the second conjugation problem without parameters:

$$\frac{\partial u_j}{\partial n_{jk4}}=-\frac{\partial u_k}{\partial n_{kj4}}=-\delta_{jk4}(\gamma_{jk4}u_j-\gamma_{kj4}u_k)\quad(\text{on }\Gamma_{jk4}).\tag{2.60}$$

Let us formulate three types of boundary conditions on free (external) boundaries.

1°. The Newton–Neumann condition with a parameter:

$$\frac{\partial u_j}{\partial n_{jj1}}+\delta_{jj1}\gamma_{jj1}u_j=\mu\alpha_{jj1}\gamma_{jj1}u_j\quad(\text{on }\Gamma_{jj1}).\tag{2.61}$$

2°. The Newton–Neumann condition without parameters:

$$\frac{\partial u_j}{\partial n_{jj2}}+\delta_{jj2}\gamma_{jj2}u_j=0\quad(\text{on }\Gamma_{jj2}).\tag{2.62}$$

3°. The Dirichlet condition:

$$\gamma_{jj3}u_j=0\quad(\text{on }\Gamma_{jj3}).\tag{2.63}$$

Here the operators α_{jjl} and δ_{jjl}, $l=1,2$, have the same general properties (see (2.56) and (2.57)) as above, i.e., as $k>j$.

Note that in multicomponent problem (2.52)–(2.63), one of the parameters, i.e., λ or μ, is fixed and the other is spectral (see [52]). Moreover, we can impose any other homogeneous conditions of conjugation on adjoining parts of boundary and on free boundaries. In particular, if there are no conditions in the problem under consideration, then we can accept that the measure of corresponding parts of the boundary is zero.

Let us give the statement of an abstract multicomponent problem which bears on the abstract Green formula (2.10) for mixed boundary-value problems.

Consider the set of spaces E_j, F_j, and G_j and trace operators $\gamma_j,\ j=\overline{1,q}$, such that, for every set, the conditions of Theorem 1.1 hold and hence for every jth set, there are q abstract Green formulas of type (1.6):

$$\langle \eta_j, L_j u_j\rangle_{E_j} = (\eta_j, u_j)_{F_j} - \langle \gamma_j\eta_j, \partial_j u_j\rangle_{G_j} \quad \forall \eta_j, u_j \in F_j,\ j=\overline{1,q}. \tag{2.64}$$

Let Lemma 2.2 hold for any set of jth spaces.

As in the above-mentioned example, we accept the following:

1°. The following direct decompositions take place:

$$(G_+)_j = \sum_{k=1}^{q}(\dot{+})(G_+)_{jk}, \quad j=\overline{1,q}, \tag{2.65}$$

$$(G_+)_{jk} = \sum_{l}(\dot{+})(G_+)_{jkl}, \quad l=\overline{1,4}\ (k>j), \quad l=\overline{1,3}\ (k=j), \tag{2.66}$$

where we denote by $(G_+)_{jkl}$ the image of the corresponding operator γ_{jkl} given on F_j (see below).

2°. Every space from the decomposition (2.65) and (2.66) has an equipment, i.e.,

$$(G_+)_{jkl} \hookrightarrow G_{jkl} \hookrightarrow (G_+)^*_{jkl} \quad \forall j,k,l. \tag{2.67}$$

3°. Equipments (2.67) are equal after we move indices j and k, i.e.,

$$(G_+)_{jkl} = (G_+)_{kjl}, \quad G_{jkl} = G_{kjl}, \quad (G_+)^*_{jkl} = (G_+)^*_{kjl}. \tag{2.68}$$

Let $\rho_{jkl}:\ (G_+)_j \to (G_+)_{jkl}$ be the restriction operators (from $(G_+)_j$ to the space $(G_+)_{jkl}$.) Their properties are the same as in Lemma 2.2. Let $\omega_{jkl}:\ (G_+)_{jkl} \to (\widetilde{G}_+)_{jkl}$ be the corresponding operators of extension by zero (from $(G_+)_{jkl}$ to $(\widetilde{G}_+)_{jkl} \subset (G_+)_j$.)

Then

$$\gamma_{jkl} := \rho_{jkl}\gamma_j:\ F_j \to (G_+)_{jkl}, \quad \partial_{jkl} := \omega^*_{jkl}\partial_j:\ F_j \to (G_+)^*_{jkl} \tag{2.69}$$

are the corresponding bounded abstract trace operators (to a part of the boundary) and normal derivative operators defined at a part of the boundary.

Taking into account our notation, we formulate a statement on an abstract conjugation problem. It is required to find a set of elements $\{u_j\}_{j=1}^q,\ u_j \in F_j$ such that the following equations are satisfied:

$$L_j u_j + \lambda a_j u_j = 0, \quad j=\overline{1,q}, \tag{2.70}$$

where $\lambda \in \mathbb{C}$ is a parameter and $a_j \in \mathcal{L}\left(F_j, F_j^*\right)$ are linear, bounded, positive-definite operators:

$$\langle u_j, a_j u_j\rangle_{E_j} \geq c_j \|u_j\|^2_{E_j}, \quad c_j>0, \ \ j=\overline{1,q}. \tag{2.71}$$

The solutions of Eqs. (2.70) must satisfy the following abstract boundary conditions.

We have for $k>j$ (see (2.54)–(2.60)):

1°. The conditions of the first conjugation problem with a parameter:

$$\gamma_{jk1}u_j = \gamma_{kj1}u_k, \quad \partial_{jk1}u_j + \partial_{kj1}u_k + \delta_{jk1}\gamma_{jk1}u_j = \mu\alpha_{jk1}\gamma_{jk1}. \tag{2.72}$$

Here and below, α_{jkl} and δ_{jkl}, $l=\overline{1,4}$, are the operators from $\mathcal{L}\left((G_+)_{jkl}, (G_+)^*_{jkl}\right)$ with properties that have the form of (2.56), (2.57), i.e.,

$$\langle \gamma_{jkl}u_j, \alpha_{jkl}\gamma_{jkl}u_j\rangle_{G_{jkl}} \geq c_{jkl}\|\gamma_{jkl}u_j\|^2_{G_{jkl}}, \quad c_{jkl}>0, \tag{2.73}$$

$$\langle \gamma_{jkl}u_j, \delta_{jkl}\gamma_{jkl}u_j\rangle_{G_{jkl}} \geq 0. \tag{2.74}$$

2°. The conditions of the first conjugation problem without parameters:

$$\gamma_{jk2}u_j = \gamma_{kj2}u_k, \quad \partial_{jk2}u_j + \partial_{kj2}u_k + \delta_{jk2}\gamma_{jk2}u_j = 0. \tag{2.75}$$

3°. The conditions of the second conjugation problem with a parameter:

$$\partial_{jk3}u_j = -\partial_{kj3}u_k = -\delta_{jk3}(\gamma_{jk3}u_j - \gamma_{kj3}u_k) + \mu\alpha_{jk3}(\gamma_{jk3}u_j - \gamma_{kj3}u_k). \tag{2.76}$$

4°. The conditions of the second conjugation problem without parameters:

$$\partial_{jk4}u_j = -\partial_{kj4}u_k = -\delta_{jk4}(\gamma_{jk4}u_j - \gamma_{kj4}u_k). \tag{2.77}$$

We have three types of conditions as $k = j$ (the analog of (2.61)–(2.63)).

1°. The Newton–Neumann condition with a parameter:

$$\partial_{jj1}u_j + \delta_{jj1}\gamma_{jj1}u_j = \mu\alpha_{jj1}\gamma_{jj1}u_j. \tag{2.78}$$

2°. The Newton–Neumann condition without parameters:

$$\partial_{jj2}u_j + \delta_{jj2}\gamma_{jj2}u_j = 0. \tag{2.79}$$

3°. The Dirichlet condition:

$$\gamma_{jj3}u_j = 0. \tag{2.80}$$

As above (for $k > j$), the operators α_{jjl} and δ_{jjl}, $l = 1, 2$ are positive definite and nonnegative (see (2.73) and (2.74)).

We will study problem (2.70)–(2.80) with help of abstract Green formulas for the set of jth spaces, $j = \overline{1, q}$.

Let us write the Green formulas (2.64) taking into account properties (2.65)–(2.68) and Lemma 2.2. We have

$$\langle \eta_j, L_j u_j \rangle_{E_j} = (\eta_j, u_j)_{F_j} - \sum_{k>j}\sum_{l=1}^{4} \{\langle \gamma_{jkl}\eta_j, \partial_{jkl}u_j\rangle_{G_{jkl}} + \langle \gamma_{kjl}\eta_k, \partial_{kjl}u_k\rangle_{G_{jkl}}\} - \sum_{l=1}^{3} \langle \gamma_{jjl}\eta_j, \partial_{jjl}u_j\rangle_{G_{jjl}} \quad \forall \eta_j, u_j \in F_j. \tag{2.81}$$

Summing the left-hand sides and right-hand sides over j from 1 to q, we obtain the Green formula

$$\sum_{j=1}^{q} \langle \eta_j, L_j u_j \rangle_{E_j} = \sum_{j=1}^{q} (\eta_j, u_j)_{F_j} - \sum_{j=1}^{q}\sum_{k>j}\sum_{l=1}^{4} \{\langle \gamma_{jkl}\eta_j, \partial_{jkl}u_j\rangle_{G_{jkl}} + \langle \gamma_{kjl}\eta_k, \partial_{kjl}u_k\rangle_{G_{jkl}}\} - \sum_{j=1}^{q}\sum_{l=1}^{3} \langle \gamma_{jjl}\eta_j, \partial_{jjl}u_j\rangle_{G_{jjl}}, \tag{2.82}$$

$$\eta := (\eta_1, \ldots, \eta_q),\ u := (u_1, \ldots, u_q) \in F := \bigoplus_{j=1}^{q} F_j. \tag{2.83}$$

3. Auxiliary Abstract Boundary-Value Problems and Representations of Solutions

Abstract boundary-value problems generated by problem (2.70)–(2.80) and operators of these boundary-value problems are studied with help of the Green formula (2.82). Two of such problems sometimes called auxiliary problems of S. G. Krein, will be considered. They are used in self-adjoint and non self-adjoint problems of mathematical physics, in particular, in hydrodynamics and theory of elasticity (see, e.g., [31, Secs. 7-8].)

3.1. Preliminary transformations. Let us consider a priori properties of problem (2.70)–(2.80). We can divide boundary conditions (2.72)–(2.80) into two classes as is customary for the calculus of variations. Let us refer so-called *main boundary conditions* to the first class. They are the first in relations (2.72) and (2.75), and conditions (2.80). Refer *natural boundary conditions* to the second class; here they are the rest in (2.72)–(2.80).

Let us consider in the space

$$F = \bigoplus_{j=1}^{q} F_j, \quad (\eta, u)_F := \sum_{j=1}^{q} (\eta_j, u_j)_{F_j}, \ \eta, u \in F, \tag{3.1}$$

the collection V of sets $u = (u_1, \ldots, u_q)$ that satisfy the following boundary conditions:

$$V := \big\{u = (u_1, \ldots, u_q) \in F: \ \gamma_{jk1}u_j = \gamma_{kj1}u_k, \gamma_{jk2}u_j = \gamma_{kj2}u_k \ (k > j); \ \gamma_{jj3}u_j = 0, \ j = \overline{1, q}\big\}. \tag{3.2}$$

Since all of the operators $\gamma_{jkl} : F_j \to (G_+)_{jkl}$ are bounded, we have that V is a subset of the set F.

The further constructions are based on the following assumption. As a rule, it is satisfied in problems of mathematical physics. Let every subset $N_j := \ker \gamma_j$ be dense in E_j, i.e.,

$$\overline{N_j} = E_j, \quad j = \overline{1, q}. \tag{3.3}$$

For example, if $j = 1$ and $F = H^1(\Omega), \ E = L_2(\Omega), \ \gamma$ is a trace operator at $\Gamma = \partial\Omega, \ \Omega \subset \mathbb{R}^m$, then

$$N := \ker \gamma = H_0^1(\Omega), \quad \overline{H_0^1(\Omega)} = L_2(\Omega). \tag{3.4}$$

Lemma 3.1. *The following orthogonal decomposition holds*:

$$F = V \oplus V^{\perp}, \tag{3.5}$$

$$V^{\perp} := \big\{u = (u_1, \ldots, u_q) \in F: \ L_j u_j = 0, \ \partial_{jjl}u_j = 0, \ l = 1, 2,$$
$$\partial_{jkl}u_j + \partial_{kjl}u_k = 0, \ k > j, \ l = \overline{1, 4}, \ j = \overline{1, q}\big\}. \tag{3.6}$$

Proof. Let $\eta := (\eta_1, \ldots, \eta_q) \in V$, and $u \in F$ be orthogonal to η with respect to the inner product in the space F. Then, according to formulas (2.82) and (3.2) we have

$$\sum_{j=1}^{q} \langle \eta_j, L_j u_j \rangle_{E_j} + \sum_{j=1}^{q} \sum_{k>j} \sum_{l=1}^{4} \langle \gamma_{jkl}\eta_j, \partial_{jkl}u_j + \partial_{kjl}u_k \rangle_{G_{jkl}} + \sum_{j=1}^{q} \sum_{l=1}^{2} \langle \gamma_{jjl}\eta_j, \partial_{jjl}u_j \rangle_{G_{jjl}} = 0. \tag{3.7}$$

Setting here $\eta = (\eta_1, 0, \ldots, 0), \ \eta_1 \in N_1$, we see, taking into account property (3.3), that $L_1 u_1 = 0$. Similarly, we obtain that

$$L_j u_j = 0, \quad j = \overline{1, q}. \tag{3.8}$$

Then the first sum in (3.7) is equal to zero.

If all $\gamma_{jkl}\eta_j = 0$ $(k > j)$, then the second sum in (3.7) is equal to zero, and as the collection of elements $\{\gamma_{jjl}\eta_j\}_{\eta_j \in F_j}$ covers all the space $(G_+)_{jjl}$, we obtain the following conditions:

$$\partial_{jjl}u_j = 0, \quad j = \overline{1, q}, \ l = 1, 2. \tag{3.9}$$

It follows from (3.7) that the second sum in (3.7) is equal to zero.

Finally, using the fact that $\{\gamma_{jkl}\eta_j\}_{\eta_j \in F_j}$ covers all $(G_+)_{jkl}$, we obtain that

$$\partial_{jkl}u_j + \partial_{kjl}u_k = 0, \quad k > j, \ j = \overline{1, q}, \ l = \overline{1, 4}. \tag{3.10}$$

The statement of the lemma follows from here. □

It follows from Lemma 3.1 that the subset $V^{\perp}$ from (3.5) consists of those sets of harmonic elements (see (3.8)) that satisfy boundary conditions (3.9) and (3.10).

To illustrate decomposition (3.5) let us consider a simple example. Let us have at the plane $\mathbb{R}^2$ three domains $\Omega_1, \ \Omega_2$ and Ω_3, having Lipschitz boundaries $\Gamma_1, \ \Gamma_2$ and Γ_3, respectively. Let them be adjoint by parts of these boundaries $\Gamma_{12} = \Gamma_{21},\ \Gamma_{13} = \Gamma_{31},\ \Gamma_{23} = \Gamma_{32}$ and have free parts $\Gamma_{11},\ \Gamma_{22}$ and Γ_{33}, respectively.

Let us introduce spaces $H^1(\Omega_1),\ H^1(\Omega_2)$ and $H^1(\Omega_3)$ having standard norms and spaces $L_2(\Omega_1), L_2(\Omega_2)$ and $L_2(\Omega_3)$. Let us form the space $F := \bigoplus_{j=1}^{3} H^1(\Omega_j)$ and introduce its subspace V of elements $u = (u_1; u_2; u_3)$ of the following form (see (3.2)):

$$V := \big\{u = (u_1, u_2, u_3) \in F :\ u_1 = u_2 \text{ (at } \Gamma_{12}),\ u_2 = u_3 \text{ (at } \Gamma_{23}),$$
$$u_1 = u_3 \text{ (at } \Gamma_{13}),\ u_1 = 0 \text{ (at } \Gamma_{11}),\ u_2 = 0 \text{ (at } \Gamma_{22}),\ u_3 = 0 \text{ (at } \Gamma_{33})\big\}.$$

Then by virtue of Lemma 3.1 we have orthogonal decomposition (3.5) where

$$V^{\perp} := \big\{u = (u_1, u_2, u_3) \in F :\ u_j - \Delta u_j = 0 \text{ (in } \Omega_j),\ j = 1, 2, 3,$$
$$\frac{\partial u_1}{\partial n_{12}} + \frac{\partial u_1}{\partial n_{21}} = 0 \text{ (on } \Gamma_{12}),\ \frac{\partial u_2}{\partial n_{23}} + \frac{\partial u_3}{\partial n_{31}} = 0 \text{ (on } \Gamma_{23}),\ \frac{\partial u_3}{\partial n_{31}} + \frac{\partial u_1}{\partial n_{13}} = 0 \text{ (on } \Gamma_{13})\big\}.$$

Here $\vec{n}_{ij}$ is an external inward normal to $\Omega_j,\ \Omega_j,\ j = 1, 2, 3$.

Let us use the Green formula for the elements from V. It can be derived from (2.82) and boundary conditions (3.2). We have

$$\sum_{j=1}^{q} \langle \eta_j, L_j u_j \rangle_{E_j} = \sum_{j=1}^{q} (\eta_j, u_j)_{F_j} - \sum_{j=1}^{q} \sum_{k>j} \sum_{l=1}^{2} \langle \gamma_{jkl}\eta_j, \partial_{jkl} u_j + \partial_{kjl} u_k \rangle_{G_{jkl}}$$
$$- \sum_{j=1}^{q} \sum_{k>j} \sum_{l=3}^{4} \big\{\langle \gamma_{jkl}\eta_j, \partial_{jkl} u_j \rangle_{G_{jkl}} + \langle \gamma_{kjl}\eta_k, \partial_{kjl} u_k \rangle_{G_{jkl}}\big\} - \sum_{j=1}^{q} \sum_{l=1}^{2} \langle \gamma_{jjl}\eta_j, \partial_{jjl} u_j \rangle_{G_{jjl}}. \quad (3.11)$$

Let us extract the following subspace of V:

$$N := \bigoplus_{j=1}^{q} N_j \subset V \subset F \subset E := \bigoplus_{j=1}^{q} E_j. \quad (3.12)$$

By virtue of (3.3) we have properties of density

$$\overline{N} = \overline{V} = E. \quad (3.13)$$

Lemma 3.2. *The following orthogonal decomposition takes place in the inner product of the space F:*

$$V = N \oplus N^{\perp}, \quad (3.14)$$

$$N^{\perp} := \big\{u = (u_1, \dots, u_q) \in V :\ L_j u_j = 0,\ j = \overline{1, q}\big\}. \quad (3.15)$$

Proof. It follows directly from formulas (3.11) and (3.3) taking into account the fact that for elements from N every $\gamma_{jkl} u_j = 0$. □

Let $u = (u_1, \dots, u_q) \in V$ be a solution of problem (2.70)–(2.80). Then for every $\eta \in V$ we have the following equation using the Green formula (3.11):

$$
\begin{aligned}
(\eta, u)_V &:= \sum_{j=1}^{q} (\eta_j, u_j)_{F_j} + \sum_{j=1}^{q} \sum_{k>j} \sum_{l=1}^{2} \langle \gamma_{jkl}\eta_j, \delta_{jkl}\gamma_{jkl}u_j \rangle_{G_{jkl}} \\
&+ \sum_{j=1}^{q} \sum_{k>j} \sum_{l=3}^{4} \langle (\gamma_{jkl}\eta_j - \gamma_{kjl}\eta_k), \delta_{jkl}(\gamma_{jkl}u_j - \gamma_{kjl}u_k) \rangle_{G_{jkl}} + \sum_{j=1}^{q} \sum_{l=1}^{2} \langle \gamma_{jjl}\eta_j, \delta_{jjl}\gamma_{jjl}u_j \rangle_{G_{jjl}} \\
&= -\overline{\lambda} \sum_{j=1}^{q} \langle \eta_j, a_j u_j \rangle_{E_j} + \overline{\mu} \Big\{ \sum_{j=1}^{q} \sum_{k>j} \big[\langle \gamma_{jk1}\eta_j, \alpha_{jk1}\gamma_{jk1}u_j \rangle_{G_{jk1}} \\
&+ \langle (\gamma_{jk3}\eta_j - \gamma_{kj3}\eta_k), \alpha_{jk3}(\gamma_{jk3}u_j - \gamma_{kj3}u_k) \rangle_{G_{jk3}} \big] \\
&+ \sum_{j=1}^{q} \langle \gamma_{jj1}\eta_j, \alpha_{jj1}\gamma_{jj1}u_j \rangle_{G_{jj1}} \Big\} =: -\overline{\lambda}\Phi_1(\eta, u) + \overline{\mu}\Phi_2(\eta, u).
\end{aligned} \tag{3.16}
$$

3.2. The first auxiliary boundary-value problem. Let us introduce in the space V the inner product defined by the left-hand side of (3.16) and the corresponding norm.

Lemma 3.3. *The norms defined by the inner products* (3.1) *and* (3.16) *are equivalent.*

Proof. Since all the operators δ_{jkl}, $l = \overline{1,4}$, are nonnegative, we see that $\|\eta\|_V \geq \|\eta\|_F$ for every $\eta \in V$. Besides, as the operators γ_{jkl} are bounded from F_j to $(G_+)_{jkl}$, and δ_{jkl} is bounded from $(G_+)_{jkl}$ to $(G_+)^*_{jkl}$ we have that the corresponding sums of quadratic functionals of the expression $(\eta, \eta)_V$ (see (3.16)) can be estimated by the value $c\|\eta\|^2_F$, $c > 0$. Therefore,

$$\|\eta\|_F \leq \|\eta\|_V \leq (1 + c)^{1/2}\|\eta\|_F. \tag{3.17}$$

□

Using these facts let us consider the first auxiliary boundary-value problem generated by problem (2.70)–(2.80). It can be formally obtained by replacing $-\lambda a_j u_j$ by f_j in (2.70) as $\mu = 0$ in (2.72), (2.76), and (2.78).

The problem called the *first S. G. Krein auxiliary problem* arises:

$$
\begin{aligned}
&L_j v_j = f_j, \quad j = \overline{1,q}; \\
&\gamma_{jkl}v_j = \gamma_{kjl}v_k, \quad \partial_{jkl}v_j + \partial_{kjl}v_k + \delta_{jkl}\gamma_{jkl}v_j = 0, \quad l = 1,2; \\
&\partial_{jkl}v_j = -\partial_{kjl}v_k = -\delta_{jkl}(\gamma_{jkl}v_j - \gamma_{kjl}v_k), \quad l = 3,4, \quad k > j; \\
&\partial_{jjl}v_j + \delta_{jjl}\gamma_{jjl}v_j = 0, \quad l = 1,2; \\
&\gamma_{jj3}v_j = 0, \quad j = \overline{1,q}.
\end{aligned} \tag{3.18}
$$

Definition 3.1. Let us say that an element $v := (v_1, \dots, v_q) \in V$ is a *weak* (*variation*) solution of problem (3.18) if the following equation holds:

$$\langle \eta, f \rangle_E := \sum_{j=1}^{q} \langle \eta_j, f_j \rangle_{E_j} = (\eta, v)_V \quad \forall \eta \in V. \tag{3.19}$$

Equation (3.19) can be derived from Eqs. (3.18) and the Green formula (3.11) for the elements η and v.

Theorem 3.1. *Assume that*

$$f = (f_1, \dots, f_q) \in V^*. \tag{3.20}$$

Then the following assertions hold:

1°. *The operator form* (3.18) *and variation form* (3.19) *of the first S. G. Krein problem are equivalent.*

2°. *Since*

$$f = (f_1, \dots, f_q) \in E = \bigoplus_{j=1}^{q} E_j,$$

problem (3.18) *has a unique general solution*

$$v = A^{-1} f \in \mathcal{D}(A) \subset V,$$

where A is a self-adjoint, positive definite operator, which is a generating operator for the pair of spaces $(V; E)$. Moreover, the following equation holds:

$$(\eta, Av)_E = (\eta, v)_V = (A^{1/2}\eta, A^{1/2}v)_E \quad \forall \eta \in V,\ v \in \mathcal{D}(A). \tag{3.21}$$

3°. *Under condition* (3.20), *problem* (3.19) *has a unique weak solution*

$$v = A^{-1} f \in V, \tag{3.22}$$

for which the following equation holds:

$$\langle \eta, Av \rangle_E = (\eta, v)_V = (A^{1/2}\eta, A^{1/2}v)_E \quad \forall \eta \in V. \tag{3.23}$$

Vice versa, each element $v \in V$ is a weak solution of problem (3.18) *as $f = Av \in V^*$.*

Proof. The proof is standard and is based on the fact that the left-hand side in (3.19) under the condition (3.20) is a bounded linear functional with respect to $\eta \in V$. Moreover, the inner product in V is given by formula (3.16). Therefore, we take into account the equivalence property of the norms (3.17).

As was already mentioned, Eq. (3.19) follows from Eq. (3.18). Let us prove the inverse assertion. From (3.19), the Green formula (3.11), and the definition (3.16) of the inner product in V, we have

$$\begin{aligned}
\sum_{j=1}^{q} \langle \eta_j, L_j v_j - f_j \rangle_{E_j} &+ \sum_{j=1}^{q} \sum_{k>j} \sum_{l=1}^{2} \langle \gamma_{jkl}\eta_j, \partial_{jkl} v_j + \partial_{kjl} v_k + \delta_{jkl}\gamma_{jkl} v_j \rangle_{G_{jkl}} \\
&+ \sum_{j=1}^{q} \sum_{k>j} \sum_{l=3}^{4} \langle \gamma_{jkl}\eta_j, \partial_{jkl} v_j + \delta_{jkl}(\gamma_{jkl} v_j - \gamma_{kjl} v_k) \rangle_{G_{jkl}} \\
+ \sum_{j=1}^{q} \sum_{k>j} \sum_{l=3}^{4} \langle \gamma_{kjl}\eta_k, \partial_{kjl} v_k - \delta_{jkl}(\gamma_{jkl} v_j - \gamma_{kjl} v_k) \rangle_{G_{jkl}} &+ \sum_{j=1}^{q} \sum_{l=1}^{2} \langle \gamma_{jjl}\eta_j, \partial_{jjl} v_j + \delta_{jjl}\gamma_{jjl} v_j \rangle_{G_{jjl}} = 0.
\end{aligned} \tag{3.24}$$

Now we use the same reasoning as in the proof of Lemma 3.1. We set $\eta \in N$, and assume that the first conditions of (3.18) holds, i.e., $L_j v_j = f_j$. Assuming that only $\gamma_{jjl}\eta_j$ has arbitrary values from $(G_+)_{jjl}$, and the other components of the element $\eta \in V$ are zero, we conclude that $\partial_{jjl} v_j + \delta_{jjl}\gamma_{jjl} v_j = 0$, $l = 1, 2$, and these relations are valid in $(G_+)^*_{jjl}$.

Reasoning similarly with respect to the second, third, and fourth sums of (3.24) we see that the other natural (from the variation point of view) boundary conditions from (3.18) hold. Moreover, all the summands from these relations are elements of the corresponding spaces $(G_+)^*_{jkl}$.

The other statements of the theorem can be proved with help of a standard scheme (see, e.g., [31, pp. 32–42].) □

3.3. On the abstract Green formula for multicomponent conjugation problems. Let us transform the Green formula (3.11) by extracting the inner product $(\eta, u)_V$ from (3.16) in explicit form. This gives the equation

$$\sum_{j=1}^{q}\langle \eta_j, L_j u_j\rangle_{E_j} = (\eta, u)_V - \Big\{ \sum_{j=1}^{q}\sum_{k>j}\sum_{l=1}^{2}\langle \gamma_{jkl}\eta_j, \partial_{jkl}u_j + \partial_{kjl}u_k + \delta_{jkl}\gamma_{jkl}u_j\rangle_{G_{jkl}}$$
$$+ \sum_{j=1}^{q}\sum_{k>j}\sum_{l=3}^{4}\langle \gamma_{jkl}\eta_j, \partial_{jkl}u_j + \delta_{jkl}(\gamma_{jkl}u_j - \gamma_{kjl}u_k)\rangle_{G_{jkl}}$$
$$+ \sum_{j=1}^{q}\sum_{k>j}\sum_{l=3}^{4}\langle \gamma_{kjl}\eta_k, \partial_{kjl}u_k - \delta_{jkl}(\gamma_{jkl}u_j - \gamma_{kjl}u_k)\rangle_{G_{jkl}} + \sum_{j=1}^{q}\sum_{l=1}^{2}\langle \gamma_{jjl}\eta_j, \partial_{jjl}u_j + \delta_{jjl}\gamma_{jjl}u_j\rangle_{G_{jjl}}\Big\} \tag{3.25}$$

for every $\eta, u \in V$.

Let us introduce the following notation for the elements η and u from V:

$$Lu := (L_1u_1, \dots, L_qu_q) \in V^*, \tag{3.26}$$

$$\gamma\eta := \big(\gamma_{jkl}\eta_j,\ l=1,2,\ k>j,\ j=\overline{1,q};\quad \gamma_{jkl}\eta_j,\ l=3,4,\ k>j,\ j=\overline{1,q};$$
$$\gamma_{kjl}\eta_k,\ l=3,4,\ k>j,\ j=\overline{1,q};\quad \gamma_{jjl}\eta_j,\ l=1,2,\ j=\overline{1,q}\big)$$

$$\in G_+ := \left(\sum_{j=1}^{q}\sum_{k>j}\sum_{l=1}^{2}(\dot{+})(G_+)_{jkl}\right)$$
$$(\dot{+})\left(\sum_{j=1}^{q}\sum_{k>j}\sum_{l=3}^{4}(\dot{+})\Big((G_+)_{jkl}(\dot{+})(G_+)_{jkl}\Big)\right)(\dot{+})\left(\sum_{j=1}^{q}\sum_{l=1}^{2}(\dot{+})(G_+)_{jjl}\right)$$
$$\subsetneq G := \left(\sum_{j=1}^{q}\sum_{k>j}\sum_{l=1}^{2}\oplus G_{jkl}\right)\oplus\left(\sum_{j=1}^{q}\sum_{k>j}\sum_{l=3}^{4}\oplus\big(G_{jkl}\oplus G_{jkl}\big)\right)\oplus\left(\sum_{j=1}^{q}\sum_{l=1}^{2}\oplus G_{jjl}\right); \tag{3.27}$$

$$\partial u := \big(\partial_{jkl}u_j + \partial_{kjl}u_k + \delta_{jkl}\gamma_{jkl}u_j,\ l=1,2,\ k>j,\ j=\overline{1,q};$$
$$\partial_{jkl}u_j + \delta_{jkl}(\gamma_{jkl}u_j - \gamma_{kjl}u_k),\ l=3,4,\ k>j,\ j=\overline{1,q};$$
$$\partial_{kjl}u_k - \delta_{jkl}(\gamma_{jkl}u_j - \gamma_{kjl}u_k),\ l=3,4,\ k>j,\ j=\overline{1,q};\quad \partial_{jjl}u_j + \delta_{jjl}\gamma_{jjl}u_j,\ l=1,2,\ j=\overline{1,q}\big)$$
$$\in (G_+)^* := \left(\sum_{j=1}^{q}\sum_{k>j}\sum_{l=1}^{2}(\dot{+})(G_+)^*_{jkl}\right)(\dot{+})\left(\sum_{j=1}^{q}\sum_{k>j}\sum_{l=3}^{4}(\dot{+})\Big((G_+)^*_{jkl}(\dot{+})(G_+)^*_{jkl}\Big)\right)$$
$$(\dot{+})\left(\sum_{j=1}^{q}\sum_{l=1}^{2}(\dot{+})(G_+)^*_{jjl}\right),\quad G_+ \subsetneq G \subsetneq (G_+)^*. \tag{3.28}$$

Then formula (3.25) can be rewritten in brief form as follows:

$$\langle \eta, Lu\rangle_E = (\eta, u)_V - \langle \gamma\eta, \partial u\rangle_G \quad \forall \eta, u \in V, \tag{3.29}$$

where $\langle \gamma\eta, \partial u\rangle_G$ is the expression in the braces from (3.25). In this form, it is the same as the Green formula for a set of spaces $E,\ V$ and G and for the trace operator γ (see (1.6)).

Recall that here η and u are the sets of elements of the type of (3.2) satisfying the main boundary conditions. Let us note that if all $\delta_{jkl} = 0$, then

$$(\eta, u)_V = (\eta, u)_F = \sum_{j=1}^{q} (\eta_j, u_j)_{F_j}, \tag{3.30}$$

and the Green formula (3.25) turns into formula (3.11).

Taking into account notation (3.26)–(3.28), we can rewrite the first auxiliary problem (3.18) in the following form:

$$Lv = f, \quad \partial v = 0. \tag{3.31}$$

Consequently, the operator A of the Hilbert pair $(V; E)$ is defined on the following domain:

$$\mathcal{D}(A) = \{v \in V : \ Lv \in E, \ \partial v = 0\} \subset V. \tag{3.32}$$

After the extension of this operator to $V = \mathcal{D}(A^{1/2})$, we have

$$\mathcal{D}(A) = V, \quad \mathcal{R}(A) = V^*, \quad A^{1/2}V = E, \quad A^{1/2}E = V^*. \tag{3.33}$$

3.4. The second auxiliary boundary-value problem. Let us consider a boundary-value problem for homogeneous equations and nonhomogeneous boundary conditions. Such a problem can be formally obtained from (2.70)–(2.80) as $\lambda = 0$ and after substituting all the natural boundary conditions for the corresponding nonhomogeneous conditions.

The problem called the *second auxiliary S. G. Krein problem* arises:

$$\begin{aligned} L_j w_j = 0, &\qquad j = \overline{1,q}; \\ \gamma_{jkl} w_j = \gamma_{kjl} w_k, \quad \partial_{jkl} w_j + \partial_{kjl} w_k + \delta_{jkl}\gamma_{jkl} w_j = \psi_{jkl}, &\qquad l = 1,2; \\ \partial_{jkl} w_j + \delta_{jkl}(\gamma_{jkl} w_j - \gamma_{kjl} w_k) = \psi_{jkl}, &\qquad \\ \partial_{kjl} w_k - \delta_{jkl}(\gamma_{jkl} w_j - \gamma_{kjl} w_k) = \psi_{kjl}, &\qquad l = 3,4; \\ \partial_{jjl} w_j + \delta_{jjl}\gamma_{jjl} w_j = \psi_{jjl}, &\qquad l = 1,2. \end{aligned} \tag{3.34}$$

Here, as above, the last boundary conditions are given for $j = \overline{1,q}$, and the others are given for $k > j$, $j = \overline{1,q}$.

Using notation (3.26)–(3.28), we can rewrite problem (3.34) in brief form:

$$Lw = 0, \quad \partial w = \psi, \tag{3.35}$$

$$\begin{aligned} \psi := \big(\psi_{jkl}, \ l = 1,2, \ k > j, \ j = \overline{1,q}; \ \ \psi_{jkl}, \ l = 3,4, \ k > j, \ j = \overline{1,q}; \\ \psi_{kjl}, \ l = 3,4, \ k > j, \ j = \overline{1,q}; \ \ \psi_{jjl}, \ l = 1,2, \ j = \overline{1,q}\big). \end{aligned} \tag{3.36}$$

Definition 3.2. We say that an element $w := (w_1, \ldots, w_q) \in V$ is a *weak* (*variation*) solution of problem (3.34) (or (3.35)) if

$$\langle \gamma\eta, \psi \rangle_E = (\eta, w)_V \quad \forall \eta \in V. \tag{3.37}$$

Obviously, Eq. (3.37) follows from Eqs. (3.35) and (3.29).

Theorem 3.2. *Assume that*

$$\psi \in (G_+)^*. \tag{3.38}$$

Then the following assertions hold:

1°. *The operator form* (3.34) *and variational form* (3.37) *of the second auxiliary S. G. Krein problem are equivalent.*

2°. *Under condition* (3.38), *problem* (3.37) *has a unique weak solution*

$$w =: T_M\psi, \quad T_M: \ (G_+)^* \to M \subset V, \tag{3.39}$$

where M *is the subspace of "harmonic" elements,*

$$M := \{w = T_M\psi: \ \psi \in (G_+)^*\}. \tag{3.40}$$

Moreover, the following orthogonal decomposition holds in the inner product of the space V:

$$V = N \oplus M, \quad N = \bigoplus_{j=1}^{q} N_j, \quad N_j = \ker\gamma_j, \ j = \overline{1,q}. \tag{3.41}$$

Proof. The proof is standard, as in Theorem 3.1, and it is based on the fact that the left-hand side of Eq. (3.37) is a bounded linear functional on V iff condition (3.38) is valid.

Let us prove the first statement. It suffices to verify that all relations (3.34) follow from (3.37). Indeed, by virtue of the Green formula (3.25) and the definition of the expression $\langle\gamma\eta,\psi\rangle_E$, we have

$$\begin{aligned}(\eta,w)_V - \langle\gamma\eta,\psi\rangle_E = \sum_{j=1}^{q}\langle\eta_j, L_jw_j\rangle_{E_j} + \sum_{j=1}^{q}\sum_{k>j}\sum_{l=1}^{2}\langle\gamma_{jkl}\eta_j, \partial_{jkl}w_j + \partial_{kjl}w_k + \delta_{jkl}\gamma_{jkl}w_j - \psi_{jkl}\rangle_{G_{jkl}}\\ + \sum_{j=1}^{q}\sum_{k>j}\sum_{l=3}^{4}\langle\gamma_{jkl}\eta_j, \partial_{jkl}w_j + \delta_{jkl}(\gamma_{jkl}w_j - \gamma_{kjl}w_k) - \psi_{jkl}\rangle_{G_{jkl}}\\ + \sum_{j=1}^{q}\sum_{k>j}\sum_{l=3}^{4}\langle\gamma_{kjl}\eta_k, \partial_{kjl}w_k - \delta_{jkl}(\gamma_{jkl}w_j - \gamma_{kjl}w_k) - \psi_{kjl}\rangle_{G_{jkl}}\\ + \sum_{j=1}^{q}\sum_{l=1}^{2}\langle\gamma_{jjl}\eta_j, \partial_{jjl}w_j + \delta_{jjl}\gamma_{jjl}w_j - \psi_{jjl}\rangle_{G_{jjl}} = 0.\end{aligned}$$

The first statement follows from this since all $\gamma_{jkl}\eta_j$ are arbitrary.

Let us prove the second statement of the theorem. Since $\psi \in (G_+)^*$, the expression $\langle\gamma\eta,\psi\rangle_E$ is a bounded linear functional on V; therefore, for every $\psi \in (G_+)^*$, there exists a unique element $w =: T_M\psi \in V$ such that Eq. (3.37) for every $\eta \in V$. Moreover, $T_M: \ (G_+)^* \to V$ is a bounded linear operator. This implies that the range of values of the operator T_M forms a subspace of V, which we will call the subspace of "harmonic" elements.

Finally, let us verify that decomposition (3.41) is valid in the inner product of the space V. Let $w \in M$ and $\eta \perp M$. Then

$$\begin{aligned}(\eta,w)_V = \sum_{j=1}^{q}\langle\eta_j, L_jw_j\rangle_{E_j} + \sum_{j=1}^{q}\sum_{k>j}\sum_{l=1}^{2}\langle\gamma_{jkl}\eta_j, \partial_{jkl}w_j + \partial_{kjl}w_k + \delta_{jkl}\gamma_{jkl}w_j\rangle_{G_{jkl}} +\\ + \sum_{j=1}^{q}\sum_{k>j}\sum_{l=3}^{4}\left[\langle\gamma_{jkl}\eta_j, \partial_{jkl}w_j + \delta_{jkl}(\gamma_{jkl}w_j - \gamma_{kjl}w_k)\rangle_{G_{jkl}} + \langle\gamma_{kjl}\eta_k, \partial_{kjl}w_k - \delta_{jkl}(\gamma_{jkl}w_j - \gamma_{kjl}w_k)\rangle_{G_{jkl}}\right]\\ + \sum_{j=1}^{q}\sum_{l=1}^{2}\langle\gamma_{jjl}\eta_j, \partial_{jjl}w_j + \delta_{jjl}\gamma_{jjl}w_j\rangle_{G_{jjl}} = 0.\end{aligned} \tag{3.42}$$

Since $L_j w_j = 0$ here and the second factors in every functional have arbitrary values $\psi_{jkl} \in (G_+)^*_{jkl}$ (see (3.34)), we obtain the following relations from (3.42):

$$\gamma_{jkl}\eta_j = 0, \quad l = \overline{1,4},\ k > j,\ j = \overline{1,q};$$
$$\gamma_{kjl}\eta_j = 0, \quad l = 3,4,\ k > j,\ j = \overline{1,q};$$
$$\gamma_{jjl}\eta_j = 0, \quad l = 1,2,\ j = \overline{1,q}.$$

Along with conditions (3.2) which show that $\eta \in V$, i.e., with the conditions

$$\gamma_{jkl}\eta_j = \gamma_{kjl}\eta_k,\ l = 1,2,\ k > j,\ j = \overline{1,q}; \quad \gamma_{jj3}\eta_j = 0,\ j = \overline{1,q},$$

we have that for all $j = \overline{1,q}$, every $\gamma_{jkl}\eta_j$, $\gamma_{kjl}\eta_k$ is equal to zero as $k > j$, $l = \overline{1,4}$, and every $\gamma_{jjl}\eta_j$ is equal to zero as $l = \overline{1,3}$. Thus, $\eta \in N$. □

3.5. Representation theorem for every element of the space of solutions. Let $u \in V$ be any element of the space of solutions of (3.2). Let us show that this element can be represented by its characteristics $Lu \in V^*$ and $\partial u \in (G_+)^*$.

Theorem 3.3. *Any element $u \in V$ can be represented a unique representation if the following form:*

$$u = v + w = A^{-1}(Lu) + T_M(\partial u), \tag{3.43}$$

where v and w are the solutions of the first and second auxiliary boundary-value problems, respectively:

$$Lv = Lu \in V^*, \quad \partial v = 0, \tag{3.44}$$
$$Lw = 0, \qquad \partial w = \partial u \in (G_+)^*, \tag{3.45}$$

and $A^{-1} : V^* \to V$, $T_M : (G_+)^* \to M \subset V$ are the corresponding operators of these problems.

Proof. Since $u \in V$, we have that $Lu \in V^*$, and by virtue of Theorem 3.1, problem (3.18) (as $f = Lu$) has a unique solution $v = A^{-1}(Lu)$. Similarly, for every $u \in V$, we have $\partial u \in (G_+)^*$, and by virtue of Theorem 3.2, problem (3.45) has a unique solution $w = T_M(\partial u)$.

Let us introduce an element $\widetilde{u} := v + w = A^{-1}(Lu) + T_M(\partial u)$ and show that $\widetilde{u} = u$. Indeed, by virtue of Theorems 3.1 and 3.2, we have (taking into account Eqs. (3.44), (2.76))

$$L\widetilde{u} = Lv = Lu, \quad \partial\widetilde{u} = \partial w = \partial u.$$

This and the Green formulas (3.29) for pair of elements $(\eta; \widetilde{u})$ and $(\eta; u)$ imply that

$$\langle \eta, Lu \rangle_E = (\eta, \widetilde{u})_V - \langle \gamma\eta, \partial u \rangle_G,$$
$$\langle \eta, Lu \rangle_E = (\eta, u)_V - \langle \gamma\eta, \partial u \rangle_G.$$

Therefore,

$$(\eta, u - \widetilde{u})_V = 0 \quad \forall \eta \in V,$$

and $\widetilde{u} = u$, and representation (3.43) is proved. □

Remark 3.1. It follows from Theorem 3.3 and the orthogonal decomposition (3.41) that there is a bijection between the elements v and w from (3.43) and elements $u_N := P_N u$ and $u_M := P_M u$, where P_N and P_M are orthogonal projectors to N and M, respectively. To prove this, let us note that (see [25, 31]) in the proof of the Green formula (1.6), the operator ∂u was constructed at elements from M and N and then it was extended by linearity to the whole space $F = N \oplus M$, i.e.,

$$\partial u = \partial_M u_M + \partial_N u_N, \quad \partial_M = T_M^{-1}. \tag{3.46}$$

Taking into account decomposition (3.41), we see that a similar approach holds for Green formula (3.29). We have

$$\partial w = \partial u = \partial_M u_M + \partial_N u_N, \quad w = T_M(\partial w) = u_M + T_M \partial_N u_N. \tag{3.47}$$

Thus,

$$v = u - w = u_N + u_M - (u_M + T_M \partial_N u_N) = u_N - T_M \partial_N u_N. \tag{3.48}$$

We obtain the inverse correspondence as follows. Since $u_N \in N$, $T_M \partial_N u_N \in M$, in (3.48) we have, taking into account $w \in M$, that

$$u_N = P_N v, \quad u_M = P_M v + w. \tag{3.49}$$

4. Applications

Abstract self-adjoint and non-self-adjoint problems generated by different problems are considered in the present section. Among these problems are the spectral Stefan problem, problems of diffraction theory, problems of the type of S. G. Krein (about normal oscillations of a viscous fluid in an open vessel), a problem on the spectrum of bounded self-adjoint operators, and problems on the surface dissipation of energy. The operators of two auxiliary S. G. Krein boundary-value problems and their properties are widely used. This allows us to simplify each problem under consideration and use some known results from the theory of self-adjoint and non-self-adjoint operators, the spectral theory of operator-functions (operator sheaves), and the theory of operators in a space with indefinite metrics.

Let us assume that below all conditions from Sec. 3 that provide the existence of the abstract Green formula (3.29) for multicomponent conjugation problems are valid.

4.1. Spectral Stefan problem with the Gibbs–Thomson condition. This problem arises in the study of phase transitions of a substance (ice melting, metal fusion, etc.). Let us give its simplest linear formulation.

Consider the following equation in a domain $\Omega \subset \mathbb{R}^m$ with a piecewise boundary $\partial\Omega$:

$$\frac{\partial u}{\partial t} = \Delta u + f(t,x), \quad \Delta u := \sum_{k=1}^{m} \frac{\partial^2 u}{\partial x_k^2}, \tag{4.1}$$

where $u = u(t,x)$ is a desired function and $f(t,x)$ is a given function. Assume that the boundary $\partial\Omega$ consists of two parts: Γ and $S := \partial\Omega \setminus \Gamma$. The second desired function $\zeta = \zeta(t,x)$, $x \in \Gamma$, which characterizes smaller motions of Γ, is given at a part of $\Gamma \in C^2$. The following conditions are valid for this function:

$$u + \Delta_\Gamma \zeta = 0, \quad \frac{\partial \zeta}{\partial t} - \frac{\partial u}{\partial n} = 0, \tag{4.2}$$

where Δ_Γ is the Laplace–Beltrami operator and $\partial/\partial n$ is the derivative with respect to the outward normal $\vec{n}$ to $\partial\Omega$. Assume that the following complementary conditions hold:

$$\zeta = 0 \quad (\text{at } \partial\Gamma), \tag{4.3}$$

$$u = 0 \quad (\text{at } S). \tag{4.4}$$

Note that problem (4.1)–(4.4) is a linearized model of the one-phase problem considered at a small time interval $t \in [0;T]$. Its statement arises, e.g., from a model problem considered in [48] (see also [12]). The first of conditions (4.2) arises from the Gibbs–Thomson law and the second arises from the Stefan condition.

Let us introduce the operator $B: \mathcal{D}(B) \subset L_2(\Gamma) \to L_2(\Gamma)$,

$$B\zeta := -\Delta_\Gamma \zeta, \quad \mathcal{D}(B) := \{\zeta \in H^2(\Gamma) : \ \zeta = 0 \ (\text{on } \partial\Gamma)\}. \tag{4.5}$$

This operator is self-adjoint and is positive definite in $L_2(\Gamma)$, and its inverse operator is positive and compact.

Let us consider the homogeneous problem (4.1)–(4.4) and its solutions depending on t by the formula $\exp(-\lambda t)$, $\lambda \in \mathbb{C}$; such solutions are called normal motions of the dynamical system. Then for amplitude elements $u = u(x)$, $x \in \Omega$, and $\zeta = \zeta(x)$, $x \in \Gamma$ (taking into account (4.5)), we have the following association: $B\zeta = u$, i.e., $u = B^{-1}\zeta$. Excluding ζ, we obtain the following spectral problem for u:

$$-\Delta u = \lambda u \quad (\text{in } \Omega), \quad u = 0 \quad (\text{at } S), \quad \frac{\partial u}{\partial n} = \lambda B^{-1} u \quad (\text{at } \Gamma). \tag{4.6}$$

Let us call this problem the spectral Stefan problem with the Gibbs–Thomson condition.

Problems of this type were considered by many authors (see, e.g., [11, 16, 17]). Recently, they were considered in [13, 36, 58, 59], including the case where the right-hand side in the boundary conditions for Γ is either positive or negative.

Let us consider a generalization of problem (4.6) to the case of a multicomponent abstract conjugation problem (an abstract generalization of the modified Stefan problem in general form was considered in [36]). Let requirements (2.65)–(2.68) and the conditions providing the existence of the Green formula (2.64) be valid. Then the problem consists of Eqs. (2.70), where λ is replaced by $-\lambda$, and the boundary conditions (2.72)–(2.80), where μ is replaced by $\pm\lambda$. We assume that requirements (2.71), (2.73), and (2.74) hold for the operators a_j, δ_{jkl}, and α_{jkl}, respectively.

Let us introduce the following notation:

$$au := (a_1 u_1, \dots, a_q u_q), \tag{4.7}$$

$$\alpha\gamma u := \big(\alpha_{jk1}\gamma_{jk1}u_j,\ 0,\ \alpha_{jk3}(\gamma_{jk3}u_j - \gamma_{kj3}u_k),\ 0, \\ -\alpha_{jk3}(\gamma_{jk3}u_j - \gamma_{kj3}u_k),\ 0\ (k > j), \alpha_{jj1}\gamma_{jj1}u_j,\ 0\ (j = \overline{1, q})\big). \tag{4.8}$$

Then we can rewrite the statement of the Stefan problem in the following brief form taking into account notation (3.26)–(3.28):

$$Lu = \lambda a u, \quad \partial u = \lambda J\alpha\gamma u, \quad u \in V, \tag{4.9}$$

where the signature operator J is diagonal in decomposition (3.27), (3.28); moreover, its diagonal consists of identity operators multiplied by ± 1 (depending on the sign at λ in the boundary condition of the initial problem).

Let us use the properties of solutions of the auxiliary boundary-value problem from Secs. 3.2–3.5 to study problem (4.9) and to reduce it to a standard problem for eigenvalues. By virtue of Theorem 3.3 and formula (3.6) the solution of problem (4.9) has the following form:

$$u = A^{-1}(Lu) + T_M(\partial u) = A^{-1}(\lambda a u) + T_M(\lambda J\alpha\gamma u), \quad u \in V,$$

where A is an operator of the Hilbert pair $(V; E)$, $A \in \mathcal{L}(V, V^*)$, and T_M is a resolving operator of the second auxiliary problem (see Sec. 3.4.)

Thus, the following spectral problem arises in the space V:

$$u = \lambda\left(A^{-1}au + T_M J\alpha\gamma u\right). \tag{4.10}$$

Let us transform it to a more symmetric form. Since $V = \mathcal{D}(A^{1/2})$, we can assume that

$$u = A^{-1/2}\eta, \quad \eta \in E. \tag{4.11}$$

Then we have from (4.10)

$$A^{-1/2}\eta = \lambda\left(A^{-1}aA^{-1/2}\eta + T_M J\alpha\gamma A^{-1/2}\eta\right).$$

Each term here is an element of $V = \mathcal{D}(A^{1/2}) \subset E$; therefore, we apply the operator $A^{1/2}$ to both sides and obtain the following equation in the space E:

$$\eta = \lambda\left(\mathcal{A}\eta + \mathcal{B}\eta\right), \quad \mathcal{A} := A^{-1/2}aA^{-1/2}, \tag{4.12}$$

$$\mathcal{B} := Q^*(J\alpha)Q, \quad Q := \gamma A^{-1/2}, \quad Q^* := A^{1/2}T_M. \tag{4.13}$$

Lemma 4.1. *Let the operator a form* (4.7) *have the following property*:

$$a^{1/2} \in \mathfrak{S}_\infty(V; E), \tag{4.14}$$

i.e., it is a compact operator acting from V to E. Then the operator $\mathcal{A}$ from (4.12) *is a compact positive operator acting in E.*

Proof. Since $A^{-1/2} \in \mathcal{L}(E;V)$ and (4.14) is valid, we see that $a^{1/2}A^{-1/2} \in \mathfrak{S}_\infty(E)$. Therefore, $\mathcal{A} = A^{-1/2}aA^{-1/2} = (a^{1/2}A^{-1/2})^*(a^{1/2}A^{-1/2}) \geq 0$. Since A and a are invertible (see (2.71) and (4.7)), we have that $\mathcal{A} > 0$. □

Let us consider the properties of the operator $\mathcal{B}$ from (4.13).

Lemma 4.2. *The operators* $Q = \gamma A^{-1/2} : E \to G$ *and* $Q^* = A^{1/2}T_M : G \to E$ *are self-conjugated. If* $G_+ \hookrightarrow \hookrightarrow G$*, then these operators are compact.*

Proof. Since $\gamma : V \to G_+$ is a bounded operator and the operator $A^{-1/2}$ acts boundedly from E to V, we have that $\gamma A^{-1/2} \in \mathcal{L}(E;G_+)$. It follows, by virtue of the compactness of embedding of G_+ to G that $\gamma A^{-1/2} : E \to G$ is a compact operator.

Let us use equality (3.37) for $\eta = A^{-1/2}v,\ v \in E$ and Definition (3.39) of the operator T_M to prove the property of self conjugacy of operators Q and Q^*. We have

$$\langle \gamma A^{-1/2}v, \psi\rangle_E = (A^{-1/2}v, T_M\psi)_V = (A^{1/2}A^{-1/2}v, A^{1/2}T_M\psi)_E$$
$$= (v, A^{1/2}T_M\psi)_E \quad \forall v \in E,\ \forall \psi \in (G_+)^* \supset G.$$

It follows from here as $\psi \in G$ that

$$(v, A^{1/2}T_M\psi)_E = (\gamma A^{-1/2}v, \psi)_G \quad \forall v \in E,\ \forall \psi \in G.$$

□

Let us rewrite the statement for the quadratic form of the operator $\mathcal{B}$ using Lemma 4.2. Since $\eta = A^{1/2}u \in E$, we have by virtue of Definition (4.8) of the operator $\alpha\gamma$ and the signature operator J that (see (3.16))

$$(\eta, \mathcal{B}\eta)_E = \langle \gamma u, (J\alpha)\gamma u\rangle_G = \sum_{j=1}^{q}\sum_{k>j}\big\{ \pm \langle \gamma_{jk1}u_j, \alpha_{jk1}\gamma_{jk1}u_j\rangle_{G_{jk1}}$$
$$\pm \langle (\gamma_{jk3}u_j - \gamma_{kj3}u_k), \alpha_{jk3}(\gamma_{jk3}u_j - \gamma_{kj3}u_k)\rangle_{G_{jk3}}\big\} + \sum_{j=1}^{q}\big\{\pm\langle \gamma_{jj1}u_j, \alpha_{jj1}\gamma_{jj1}u_j\rangle_{G_{jj1}}\big\}. \quad (4.15)$$

Here the signs plus or minus are the same as in the corresponding boundary condition for the Stefan problem.

Since all the operators α_{jkl} are positive definite (see (2.73)), we have that form (4.15) is real. Assume (see property (4.14)) that the operator $\alpha \in \mathfrak{S}_\infty((G_+);(G_+)^*)$ or, equally,

$$\alpha^{1/2} \in \mathfrak{S}_\infty(G_+;G). \quad (4.16)$$

It is sufficient (by virtue of the structure of α), that all α_{jkl} from (4.15) have the following property

$$\alpha_{jkl} \in \mathfrak{S}_\infty((G_+)_{jkl};(G_+)^*_{jkl}). \quad (4.17)$$

Lemma 4.3. *If Condition* (4.16) (*or Condition* (4.17)) *holds, then the operator* $\mathcal{B}$ *from* (4.13) *is a compact self-adjoint operator acting in* E*.*

Proof. The proof follows from representation (4.13) or from form (4.15), and from Property (4.16); therefore, $\gamma A^{-1/2} \in \mathcal{L}(E;G_+)$ (Lemma 4.2). □

Lemma 4.4. *The kernel* $\ker(\mathcal{A}+\mathcal{B})$ *of the operator* $\mathcal{A}+\mathcal{B}$ *is trivial for any signature operator* J*.*

Proof. Let us consider the equation $\mathcal{A}\eta + \mathcal{B}\eta = 0$, i.e.,

$$A^{-1/2}aA^{-1/2}\eta + A^{1/2}T_M(J\alpha)\gamma A^{-1/2}\eta = 0, \quad \eta \in E.$$

Let us make the reverse substitution (4.11) and apply the operator $A^{-1/2}$ from the left. We have

$$A^{-1}au + T_M(J\alpha)\gamma u = 0, \quad u \in V. \quad (4.18)$$

It follows from solutions of the first auxiliary problem (see Theorem 3.1 and relations (3.31)–(3.33)) that (4.18) is equal to the relations

$$L\varphi = -au,\ \varphi := T_M(J\alpha)\gamma u, \quad \partial\varphi = 0.$$

Since the operator T_M acts from $(G_+)^*$ to the subspace M (see (3.39)–(3.41)), we see that $\varphi \in M$ and

$$\partial\varphi = \partial_M\varphi = \partial_M T_M(J\alpha)\gamma u = (J\alpha)\gamma u \tag{4.19}$$

as ∂_M is the left-inverse for T_M.

We have from (4.19) and (4.18) that $A^{-1}au = 0$; then as A^{-1} is positive and a is positive definite (see (4.7) and (2.71)) we can conclude that $u = 0$ and, consequently, $\eta = A^{1/2}u = 0$. □

It follows from this lemma and Lemmas 4.1 and 4.3 that the numbers $\lambda = 0$ and $\lambda = \infty$ are not eigenvalues of problem (4.12). Therefore, it is equivalent to the following problem:

$$(\mathcal{A} + \mathcal{B})\,\eta = \mu\eta, \quad \lambda = 1/\mu, \tag{4.20}$$

i.e., the problem for eigenvalues of the compact self-adjoint operator $\mathcal{A} + \mathcal{B}$.

It is known that the positive eigenvalues μ_k^+ of problem (4.20) are sequential (decreasing) positive maxima of the variation relation

$$\frac{(\mathcal{A}\eta, \eta)_E + (\mathcal{B}\eta, \eta)_E}{\|\eta\|_E^2}, \tag{4.21}$$

and the negative eigenvalues μ_k^- are sequential (increasing) negative minima of the same relation.

Lemma 4.5. *Let the conditions of Lemmas* 4.1 *and* 4.3 *hold. Then problem* (4.20) *has a branch* $\{\mu_k^+\}_{k=1}^{\infty}$ *of positive eigenvalues of finite multiplicity with a limit point* $\mu = 0$.

Proof. It is sufficient to make sure that there is an infinite-dimensional subspace of E, where functional (4.21) is positive. Nevertheless, if $\eta = A^{1/2}u,\ u \in N$, then $\gamma u = 0$ and hence $\mathcal{B}\eta = 0$. Then, by virtue of the positiveness of $\mathcal{A}$, we have that functional (4.21) gets positive values in infinite-dimensional $A^{1/2}N \subset E$. □

Note that if $J = I$, i.e., every sign at λ in the statement of problem is positive, then $\mathcal{B} \geq 0$ and, therefore, problem (4.20) has only a branch of positive eigenvalues. Let us consider the case where the signature operator J is at least once negative, i.e., $J \neq I$. This means that quadratic form (4.15) has at least one negative term. Let us show that in this case problem (4.20) has a branch $\{\mu_k^-\}_{k=1}^{\infty}$ of negative eigenvalues with a limit point zero.

It is obvious that, in our assumption, the operator $\mathcal{B}$ can be represented in the following form:

$$\mathcal{B} = \mathcal{B}_1 - \mathcal{B}_2, \quad \mathcal{B}_i \geq 0,\ i = 1, 2, \tag{4.22}$$

where $\mathcal{B}_1$ refers to a nonnegative part of quadratic form (4.15) and $\mathcal{B}_2$ refers to a negative part.

Let us consider the Steklov auxiliary problem

$$Lw = 0, \quad \partial w = \lambda(J\alpha)\gamma w, \quad w \in M. \tag{4.23}$$

Lemma 4.6. *If quadratic form* (4.15) *can be positive or negative in infinite-dimensional subspaces (and Condition* (4.16) *or* (4.17) *holds), then problem* (4.23) *has a real discrete spectrum consisting of two branches: positive eigenvalues* $\lambda_k^+ = \lambda_k^+(\mathcal{B}),\ k = 1, 2, \ldots,\ \lambda_k^+ \to +\infty\ (k \to \infty)$ *and negative eigenvalues* $\lambda_k^- = \lambda_k^-(\mathcal{B}),\ \lambda_k^- \to -\infty\ (k \to \infty)$. *The following system of eigenelements which form an orthogonal basis in subspace* $M \subset V$ *refers to these branches:* $\{w_k^+\}_{k=1}^{\infty} \cup \{w_k^-\}_{k=1}^{\infty}$.

Proof. It follows from the second S. G. Krein auxiliary problem (3.34) that Eqs. (4.23) are equal to the relation

$$w = \lambda T_M(J\alpha)\gamma w. \tag{4.24}$$

After the substitution $w = A^{-1/2}\eta,\ \eta \in E$, we obtain the equation

$$\eta = \lambda\mathcal{B}\eta = \lambda(\mathcal{B}_1 - \mathcal{B}_2)\eta, \tag{4.25}$$

where the quadratic form of operator $\mathcal{B}$ can be (it follows from the statement of the lemma) positive and negative in infinite-dimensional subspaces. Since operator $\mathcal{B}$ is compact, the statement of the lemma including the fact that eigenelements of problem (4.24) form a basis in M follows from the Hilbert–Schmidt theorem. □

It follows from Lemma 4.6 and its proof that if a quadratic form of operator $\mathcal{B}_2$ is positive in an infinite-dimensional subset, then operator $\mathcal{B}$ has a branch of negative eigenvalues $\{\mu_k^-\}_{k=1}^{\infty}$ and the sequence of eigenelements $\{w_k^-\}_{k=1}^{\infty}$ corresponding to this branch and forming an orthogonal system in $M \subset V$.

Lemma 4.7. *If the dimension of the range of values of operator $\mathcal{B}_2$ is infinite, i.e., the quadratic form* (4.15) *is negative in infinite-dimensional space (the case of $J \neq I$) and if the condition*

$$a^{1/2} \in \mathcal{L}(E) \tag{4.26}$$

and the conditions of Lemmas 4.1 *and* 4.3 *hold, then problem* (4.12) *has a branch of negative eigenvalue λ_k^-, $\lambda_k^- \to -\infty$ $(k \to \infty)$.*

Proof. It is sufficient to ensure that the quadratic functional $(\mathcal{A}\eta, \eta)_E + (\mathcal{B}\eta, \eta)_E$ is negative in the infinite-dimensional subspace.

After the substitution $\eta = A^{1/2}u$ we have

$$\begin{aligned}(\eta, \mathcal{A}\eta)_E + (\eta, \mathcal{B}\eta)_E &= (\eta, A^{-1/2}aA^{-1/2}\eta)_E + (\eta, A^{1/2}T_M(J\alpha)\gamma A^{-1/2}\eta)_E \\ &= \langle u, au\rangle_E + \langle \gamma u, (J\alpha)\gamma u\rangle_G = \|a^{1/2}u\|_E^2 + \langle \gamma u, (J\alpha)\gamma u\rangle_G. \end{aligned} \tag{4.27}$$

Let $\{w_k^-\}_{k=1}^{\infty}$ be a sequence of normalized in M eigenelements of the Steklov problem (4.24) that correspond to eigenvalues λ_k^- of this problem, $\lambda_k^- \to -\infty$ $(k \to \infty)$. Then we have, for elements of this sequence,

$$\langle \gamma w_k^-, (J\alpha)\gamma w_k^-\rangle_G = 1/\lambda_k^- < 0, \quad k = 1, 2, \ldots.$$

Since N is dense in E (see (3.13)), we have for every w_k^- and $\varepsilon_k > 0$ that there is such an element $u_k \in N$ that

$$\|w_k^- - u_k\|_E^2 < \varepsilon_k / \|a^{1/2}\|^2. \tag{4.28}$$

Quadratic form (4.27) is negative at the elements $w_k^- - u_k$, $k = 1, 2, \ldots$ because

$$\begin{aligned}&\|a^{1/2}(w_k^- - u_k)\|_E^2 + \langle \gamma(w_k^- - u_k), (J\alpha)\gamma(w_k^- - u_k)\rangle_G \\ &\qquad \le \|a^{1/2}\| \cdot \|w_k^- - u_k\|_E^2 + \langle \gamma w_k^-, (J\alpha)\gamma w_k^-\rangle_G < \varepsilon_k + 1/\lambda_k^- < 0, \quad k = 1, 2, \ldots,\end{aligned}$$

if we choose

$$0 < \varepsilon_k < -(1/\lambda_k^-), \quad k = 1, 2, \ldots.$$

Note that the sequence $\{w_k^- - u_k\}_{k=1}^{\infty}$ consists of linearly independent elements. Indeed,

$$\sum_{k=1}^{\infty} c_k(w_k^- - u_k) = 0 \quad \Rightarrow \quad \sum_{k=1}^{\infty} c_k w_k^- = \sum_{k=1}^{\infty} c_k u_k = 0,$$

since the first sum belongs to M and the second sum belongs to the orthogonal subspace N. Since the system $\{w_k^-\}_{k=1}^{\infty}$ is orthogonal in M, it follows that $c_k = 0$, $k = \overline{1, n}$.

We see that $\{w_k^- - u_k\}_{k=1}^{\infty}$ is a desired sequence of linearly independent elements from V at which quadratic form (4.27) is negative. □

Using all these facts, we formulate the final statement about properties of solutions of the multi-component abstract Stefan problem with the Gibbs–Thompson condition.

Theorem 4.1. *Let operators L, γ and ∂ be such that the abstract Green formula* (3.29) *holds. Let operators a and $\alpha\gamma$ be defined by formulas* (4.7) *and* (4.8), *and conditions* (2.71), (2.73), (2.74), (4.14), *and* (4.16) *hold. Then the following assertions hold.*

1°. *The spectral problem* (4.9) *has a discrete real spectrum consisting of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of finite multiplicity with a limit point at the infinity.*

2°. *The eigenelements $\{u_n\}_{n=1}^{\infty}$ of problem* (4.9) *corresponding to the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ form an orthonormal basis in the space V, and the following orthogonal conditions hold:*

$$(u_m, u_n)_V = \lambda_n \{\langle u_m, au_n\rangle_E + \langle \gamma u_m, (J\alpha)\gamma u_n\rangle_G\} = \delta_{mn}, \tag{4.29}$$

where δ_{mn} is the Kronecker symbol.

3°. *If $J = I$, i.e., every sign at the spectral parameter λ in the boundary conditions is positive in the statement of problem* (4.9), *then every eigenvalue of this problem is positive.*

4°. *If $J \neq I$ and, consequently, the quadratic form* (4.15) *is negative at an infinite-dimensional subspace and, moreover, if condition* (4.26) *holds, then problem* (4.9) *has two branches of eigenvalues: a positive branch $\{\lambda_k^+\}_{k=1}^{\infty}$, $\lambda_k^+ \to +\infty$ $(k \to \infty)$, and a negative branch $\{\lambda_k^-\}_{k=1}^{\infty}$, $\lambda_k^- \to -\infty$ $(k \to \infty)$.*

5°. *The following inequalities hold for the eigenvalues λ_k^+ and λ_k^-:*

$$\lambda_k^- \le \nu_k^- < 0 < \nu_k^+ \le \lambda_k^+, \quad k = 1, 2, \dots,$$

where $1/\nu_k^+$ are eigenvalues of variational relation (4.21) *for $J = I$ or the following relation* (*see* (4.27)):

$$[\langle u, au\rangle_E + \langle \gamma u, \alpha(\gamma u)\rangle_G]/\|u\|_V^2, \quad u \in V,$$

and $1/\nu_k^-$ are eigenvalues of the variational relation

$$\frac{-(B_2\eta, \eta)_E}{\|\eta\|_E^2} = \frac{-\langle \gamma u, \alpha_2(\gamma u)\rangle_G}{\|u\|_V^2}, \quad u = A^{-1/2}\eta, \ \eta \in E,$$

where α_2 is a part of the operator α that generates the operator $\mathcal{B}_2$ from (4.22) *and corresponds to negative quadratic functionals in* (4.15).

Proof. 1° follows from Lemmas 4.1–4.3 and the Hilbert–Schmidt theorem.

2° follows from the Hilbert–Schmidt theorem and the fact that $\lambda = 0$ and $\lambda = \infty$ are not eigenvalues of problem (4.12). Thus, formulas (4.29) directly follow from (4.12) after substitution $A^{-1/2}\eta = u \in V$.

3° follows from the fact that $\mathcal{A} + \mathcal{B} > 0$ as $J = I$.

4° follows from Lemma 4.7.

5° follows from the maximum principles for eigenvalues of compact self-adjoint operators. □

4.2. Conjugation problems in diffraction theory. Let us again consider multicomponent abstract conjugation problem (2.70)–(2.80) generated by the problem of diffraction. Using notation (3.26)–(3.28), and (4.7) and (4.8) we can rewrite this problem in the following short form:

$$Lu + \lambda au = 0, \quad \partial u = \mu\alpha\gamma u, \quad \lambda, \mu \in \mathbb{C}, \ u \in V. \tag{4.30}$$

Problem (4.30) can be reduced to problem (4.9) with help of substitution of λ by $-\lambda$ in the first equation and μ by λJ in the second equation. Thus, to study problem (4.30) we can use the same approaches as for problem (4.9).

Again, using representation of any element $u \in V$ (see formula (3.43)) we obtain from Eqs. (4.30) that

$$u = A^{-1}(-\lambda au) + T_M(\mu\alpha\gamma u), \quad u \in V.$$

Substituting (see (4.11)) $u = A^{-1/2}\eta$, $\eta \in E$, we obtain the following equation:

$$\eta + \lambda\mathcal{A}\eta - \mu\mathcal{B}\eta = 0, \quad \mathcal{A} = A^{-1/2}aA^{-1/2}, \ \mathcal{B} = Q^*\alpha Q, \tag{4.31}$$

where operator $\mathcal{A}$ is the same as in problem (4.12), and $\mathcal{B}$ is a special case of operator $\mathcal{B}$ from (4.13) where $J = I$.

It follows from here that the statement of Lemma 4.1 is valid for operator $\mathcal{A}$ and the statements of Lemmas 4.2–4.4 are valid for operator $\mathcal{B}$ from (4.31). Let us note that in this case the quadratic form of operator $\mathcal{B}$ has form (4.15) with the positive sign at every term, i.e.,

$$(\mathcal{B}\eta, \eta)_E = (\eta, \mathcal{B}\eta)_E = \langle \gamma u, \alpha\gamma u\rangle_G = \sum_{j=1}^{q}\sum_{k>j}\left\{\langle \gamma_{jk1}u_j, \alpha_{jk1}\gamma_{jk1}u_j\rangle_{G_{jk1}}\right.$$
$$\left. + \langle(\gamma_{jk3}u_j - \gamma_{kj3}u_k), \alpha_{jk3}(\gamma_{jk3}u_j - \gamma_{kj3}u_k)\rangle_{G_{jk3}}\right\} + \sum_{j=1}^{q}\left\{\langle \gamma_{jj1}u_j, \alpha_{jj1}\gamma_{jj1}u_j\rangle_{G_{jj1}}\right\}, \quad u = A^{-1/2}\eta. \tag{4.32}$$

Lemma 4.8. *The operator $\mathcal{B}$ has an infinite-dimensional kernel*

$$\ker\mathcal{B} = \{\eta = A^{1/2}u : \ u = (u_1, \dots, u_q) \in V, \quad \gamma_{jk1}u_j = \gamma_{kj1}u_k = 0,$$
$$\gamma_{jkl}u_j = \gamma_{kjl}u_k, \ l = 2, 3, \ k > j; \quad \gamma_{jj1}u_j = \gamma_{jj3}u_j = 0, \ j = \overline{1, q}\}. \tag{4.33}$$

Thus, the set

$$A^{1/2}N := \{\eta = A^{1/2}u : \ u \in N\} \subset \ker\mathcal{B}. \tag{4.34}$$

Proof. the proof follows from the fact that quadratic form (4.32) is zero as

$$\gamma_{jk1}u_j = 0, \quad \gamma_{jk3}u_j - \gamma_{kj3}u_k = 0, \ k > j; \quad \gamma_{jj1}u_j = 0, \ j = \overline{1, q},$$

since operators α_{jk1}, α_{jk3}, and α_{jj1} positive definite and $u \in V$. Then (see (3.2))

$$\gamma_{jk1}u_j = \gamma_{kj1}u_k, \quad \gamma_{jk2}u_j = \gamma_{kj2}u_k = 0, \ k > j; \quad \gamma_{jj3}u_j = 0, \ j = \overline{1, q}.$$

Embedding (4.34) is obvious as conditions $\gamma_{jkl}u_j = \gamma_{kjl}u_k = 0$ (for all $k \geq j$ and l) are valid for all elements from $N = \bigoplus_{j=1}^{q} N_j$. □

The following operator sheaf corresponds to Eq. (4.31):

$$\mathcal{K}(\lambda, \mu) := \mathcal{I} + \lambda\mathcal{A} - \mu\mathcal{B}.$$

It depends on the complex parameters λ and μ. Both variants are studied in diffraction problems (see [4]): either λ is fixed and μ is a spectral parameter or vice versa. These parameters are almost equivalent in problem (4.31); the difference between the variants is that $\mathcal{A}$ is positive (Lemma 4.1), and operator $\mathcal{B}$ is nonnegative (see (4.32)) and has an infinite-dimensional kernel $\ker\mathcal{B}$ (see (4.33)).

The equation of the form (4.31), i.e.,

$$\mathcal{K}(\lambda, \mu)\eta = 0, \quad \eta \in E, \tag{4.35}$$

was studied recently in papers [53, 55, 55] and [33, 34]. Therefore, the results of this study are given here without a proof.

Assume that $\lambda \in \mathbb{C}$ is fixed and μ is a spectral parameter in problem (4.35). Introduce the following notation:

$$E_0 := \ker\mathcal{B}, \quad E_1 := E \ominus E_0, \tag{4.36}$$

and denote by P_0 and P_1, respectively, the orthoprojectors on the subspaces (4.36). Representing the solution η of problem (4.35) in the form

$$\eta = \eta_0 + \eta_1 := P_0\eta + P_1\eta,$$

we obtain the following system of equations instead of (4.35):

$$\begin{aligned} &(I_0 + \lambda P_0\mathcal{A}P_0)\eta_0 + \lambda P_0\mathcal{A}P_1\eta_1 = 0, \\ &\lambda P_1\mathcal{A}P_0\eta_0 + (I_1 + \lambda P_1\mathcal{A}P_1)\eta_1 - \mu\mathcal{B}_1\eta_1 = 0, \quad \mathcal{B}_1 := P_1\mathcal{B}P_1, \end{aligned} \tag{4.37}$$

where $\mathcal{B}_1$ is a compact positive operator and I_0 and I_1 are the identity operators in E_0 and E_1, respectively.

Theorem 4.2. *Let the general condition hold in problem* (4.35) (*the non-self-adjoint case*)

$$\operatorname{Im}\lambda \neq 0. \tag{4.38}$$

Then the following statements hold:

1°. *If*

$$\mathcal{B} \in \mathfrak{S}_p(E) \tag{4.39}$$

(*where* $\mathfrak{S}_p(E)$ *is the subspace of the space of compact operators that are summable with the factor* p, *see* [19, Secs. 2, 3]); *then problem* (4.35) *has a discrete spectrum consisting of eigenvalues* $\{\mu_k(\lambda)\}_{k=1}^{\infty}$ *of finite multiplicity with a unique limit point* $\mu = \infty$. *If* $\varepsilon > 0$ *is arbitrarily small, then all eigenvalues* $\mu_k(\lambda)$, *perhaps, except for a finite number, are situated in the angle*

$$\Lambda_\varepsilon(\lambda) := \big\{\mu \in \mathbb{C} : \ |\arg \mu| < \varepsilon, \ \operatorname{sign}\operatorname{Im}\mu = \operatorname{sign}\operatorname{Im}\lambda\big\}.$$

In this case, the system of eigenelements and adjoint elements $\{\eta_k\}_{k=1}^{\infty}$ *of problem* (4.35) *corresponding to the eigenvalues* $\{\mu_k(\lambda)\}_{k=1}^{\infty}$ *after the projection to the subspace* E_1, *i.e., the system of elements* $\big\{P_1\eta_k\big\}_{k=1}^{\infty}$, *is complete in subspace* E_1.

2°. *If, instead of* (4.39), *a stronger asymptotic condition*

$$\lambda_k(\mathcal{B}) = \lambda_k(\mathcal{B}_1) = ck^{-\beta}[1 + o(1)], \quad c > 0, \ \beta > 0, \ k \to \infty$$

holds for nonzero eigenvalues of operator $\mathcal{B}$, *then the following asymptotic formula is valid for the eigenvalues* $\{\mu_k(\lambda)\}_{k=1}^{\infty}$ *of problem* (4.35):

$$\mu_k(\lambda) = \lambda_k^{-1}(\mathcal{B}_1)[1 + o(1)], \quad k \to \infty.$$

In this case, the system of elements $\big\{P_1\eta_k\big\}_{k=1}^{\infty}$ *forms the Abel–Lidsky basis* (*see* [4, p. 248-249]) *of order* $\alpha > \beta^{-1}$ *in the subspace* E_1.

If $\lambda \in \mathbb{R}$, then problem (4.35) becomes self-adjoint. The following results take place.

Theorem 4.3. *If the condition*

$$\lambda \geq 0 \tag{4.40}$$

holds (*the self-adjoint case*), *then problem* (4.35) *has a discrete spectrum* $\{\mu_k(\lambda)\}_{k=1}^{\infty}$ *consisting of positive eigenvalues of finite multiplicities*

$$\mu_k(\lambda) \geq \lambda_k^{-1}(\mathcal{B}_1), \quad k = 1, 2, \ldots,$$

with a limit point at $+\infty$.

The eigenvalues $\big\{P_1\eta_k\big\}_{k=1}^{\infty}$ *corresponding to the eigenvalues* $\{\mu_k(\lambda)\}_{k=1}^{\infty}$ *form an orthogonal basis in the energy space* $E_{F_1(\lambda)}$ *of the operator*

$$F_1(\lambda) := I_1 + \lambda P_1\mathcal{A}P_1 - \lambda^2 P_1\mathcal{A}P_0(I_0 + \lambda P_0\mathcal{A}P_0)^{-1}P_0\mathcal{A}P_1,$$

and a basis with respect to the quadratic form of the operator $\mathcal{B}_1$. *We can take basis elements satisfying the following normalization conditions*:

$$(F_1(\lambda)P_1\eta_k, P_1\eta_l) =: (P_1\eta_k, P_1\eta_l)_{F_1(\lambda)} = \delta_{kl}, \quad (\mathcal{B}_1P_1\eta_k, P_1\eta_l) = \mu_k^{-1}\delta_{kl}$$

They are equivalent to the relations

$$(\eta_k, \eta_l) + \lambda(\mathcal{A}\eta_k, \eta_l) = \delta_{kl}, \quad (\mathcal{B}\eta_k, \eta_l) = \mu_k^{-1}\delta_{kl}.$$

Definition 4.1. Let us say that the parameter $\lambda < 0$ of problem (4.37) has *nonexceptional values* if the following conditions hold:

$$1 + \lambda\lambda_k(\mathcal{A}) \neq 0, \quad 1 + \lambda\lambda_k(P_0\mathcal{A}P_0) \neq 0, \quad k = 1, 2, \dots;$$

the numbers

$$\lambda = -\lambda_k^{-1}(\mathcal{A}), \quad \lambda = -\lambda_k^{-1}(P_0\mathcal{A}P_0), \quad k = 1, 2, \dots, \tag{4.41}$$

are called exceptional values of the parameter λ.

Theorem 4.4. *Let the parameter λ have nonexceptional negative values and $\varkappa = \varkappa_{F_1(\lambda)} > 0$ be the indefiniteness range of the quadratic form of operator $F_1(\lambda)$. Then problem (4.37) for this λ has a discrete spectrum $\{\mu_k(\lambda)\}_{k=1}^{\infty}$ consisting of real eigenvalues of finite multiplicities with a unique limit point $\lambda = +\infty$. In this case, the eigenvalues $\{\mu_k(\lambda)\}_{k=1}^{\varkappa}$ are negative and the others are positive.*

The eigenelements $\{P_1\eta_k\}_{k=1}^{\infty}$ corresponding to the eigenvalues $\{\mu_k(\lambda)\}_{k=1}^{\infty}$ form a Riesz basis and an orthonormal basis in E_1 with respect to forms of operators $F_1(\lambda)$ and $\mathcal{B}_1$. We can choose elements of this basis satisfying the following normalization conditions:

$$\begin{aligned} &(F_1(\lambda)P_1\eta_k, P_1\eta_l) = -\delta_{kl}, \quad 1 \leq k, l \leq \varkappa;\\ &(F_1(\lambda)P_1\eta_k, P_1\eta_l) = \delta_{kl}, \qquad k, l > \varkappa;\\ &(F_1(\lambda)P_1\eta_k, P_1\eta_l) = 0, \qquad k \leq \varkappa,\ l > \varkappa;\\ &(\mathcal{B}_1 P_1\eta_k, P_1\eta_l) = \mu_k^{-1}(\lambda)(F_1(\lambda)P_1\eta_k, P_1\eta_l). \end{aligned}$$

The case where the parameter λ has negative exceptional values (4.41) in problem (4.37) was considered in [33, 34, 54]. It is proved that the spectrum of this problem is real and discrete with a limit point at $+\infty$, and eigenelements after projection to some infinite-dimensional subspace form a Riesz basis in this subspace.

Let us again consider problem (4.31) assuming that the parameter $\mu \in \mathbb{C}$ is fixed and λ is a spectral parameter. This problem is simpler than the previous because the operator $\mathcal{A} > 0$ is now considered as the principal operator and hence $\ker \mathcal{A} = \{0\}$. Let us state analogs of Theorems 4.2–4.4 for this case.

Theorem 4.5. *In problem (4.31) with fixed μ, let the following general condition hold:*

$$\operatorname{Im} \mu \neq 0. \tag{4.42}$$

Then the following assertions hold.

1°. *If*

$$\mathcal{A} \in \mathfrak{S}_p(E), \tag{4.43}$$

then problem (4.31) has a discrete spectrum $\{\lambda_k(\mu)\}_{k=1}^{\infty}$,consisting of eigenvalues of finite multiplicities with a unique limit point $\lambda = \infty$. For an arbitrary small $\varepsilon > 0$, all eigenvalues $\lambda_k(\mu)$, except, perhaps, a finite number, are situated in the angle

$$\Lambda_\varepsilon(\mu) := \{\lambda \in \mathbb{C} : \ |\arg(\lambda - \pi)| < \varepsilon, \ \operatorname{sign} \operatorname{Im} \mu = \operatorname{sign} \operatorname{Im} \lambda\}.$$

In this case, the system of eigenelements and adjoint elements $\{\eta_k\}_{k=1}^{\infty}$ corresponding to the eigenvalues $\{\lambda_k(\mu)\}_{k=1}^{\infty}$ is complete in the space E.

2°. *If, instead of (4.43), the following stronger asymptotic condition holds:*

$$\lambda_k(\mathcal{A}) = ck^{-\alpha}[1 + o(1)], \quad c > 0, \ \alpha > 0, \ k \to \infty,$$

then for the eigenvalues $\{\lambda_k(\mu)\}_{k=1}^{\infty}$, the following asymptotic formula is valid:

$$\lambda_k(\mu) = -\lambda_k^{-1}(\mathcal{A})[1 + o(1)], \quad k \to \infty,$$

and the system of eigenelements and adjoint elements $\{\eta_k\}_{k=1}^{\infty}$ forms the Abel–Lidsky basis of the order $\beta > \alpha^{-1}$ in the space E.

The following results are valid for a real fixed parameter μ.

Theorem 4.6. *Let $\mu \le 0$. Then problem (4.31) has a discrete spectrum $\{\lambda_k(\mu)\}_{k=1}^{\infty}$ consisting of negative eigenvalues of finite multiplicities with a limit point at $-\infty$. Eigenelements $\{\eta_k\}_{k=1}^{\infty}$ of this problem corresponding to the eigenvalues $\{\lambda_k(\mu)\}_{k=1}^{\infty}$ form an orthogonal basis in the energy space $E_{F(\mu)}$ of the operator $F(\mu) := \mathcal{I} - \mu\mathcal{B}$, and a basis with respect to the quadratic form of the operator $\mathcal{A}$. We choose basic elements satisfying the following normalization conditions*:

$$(F(\mu)\eta_k, \eta_l)_E =: (\eta_k, \eta_l)_{F(\mu)} = \delta_{kl}, \quad (\mathcal{A}\eta_k, \eta_l)_E = -\lambda_k^{-1}(\mu)\,\delta_{kl}.$$

For positive μ, we assume that

$$\mu \ne \lambda_k^{-1}(\mathcal{B}), \quad k = 1, 2, \dots, \tag{4.44}$$

where $\{\lambda_k(\mathcal{B})\}_{k=1}^{\infty}$ is a decreasing sequence of positive eigenvalues of the operator $\mathcal{B}$, $\lambda_k(\mathcal{B}) \to 0$ ($k \to \infty$); moreover,

$$1 - \lambda_k(\mathcal{B}) < 0, \ k = 1, \dots, \varkappa, \quad 1 - \lambda_{\varkappa+1}(\mathcal{B}) > 0. \tag{4.45}$$

Theorem 4.7. *Under conditions* (4.44) *and* (4.45), *problem* (4.31) *has a discrete spectrum $\{\lambda_k(\mu)\}_{k=1}^{\infty}$ consisting of positive eigenvalues of finite multiplicities with a unique limit point $\lambda = -\infty$. In this case, the eigenvalues $\{\lambda_k(\mu)\}_{k=1}^{\varkappa}$ are positive, and the others are negative.*

The eigenelements $\{\eta_k\}_{k=1}^{\infty}$ corresponding to the eigenvalues $\{\lambda_k(\mu)\}_{k=1}^{\infty}$ form a Riesz basis and an orthonormal basis with respect to the quadratic forms of the operators $F(\mu) := \mathcal{I} - \mu\mathcal{B}$ and $\mathcal{A}$. We choose basic elements satisfying the following conditions:

$$(F(\mu)\eta_k, \eta_l)_E = \begin{cases} -\delta_{kl}, & 1 \le k, l \le \varkappa; \\ 0, & k \le \varkappa, \ l > \varkappa; \\ \delta_{kl}, & k, l > \varkappa; \end{cases}$$

$$(\mathcal{A}\eta_k, \eta_l)_E = -\lambda_k^{-1}(\mu)(F(\mu)\eta_k, \eta_l)_E.$$

If the parameter $\mu > 0$ has one of the exceptional values when the equality sign is in (4.44), then statements similar to Theorem 4.7 hold. In this case, problem (4.31) has also a zero eigenvalue of finite multiplicity.

Remark 4.1. Nontrivial solutions of the Steklov problem (see (4.23))

$$Lw = 0, \quad \partial w = \mu\alpha\gamma w, \tag{4.46}$$

correspond to exceptional values of parameter μ, i.e., $\lambda_k^{-1}(\mathcal{B})$, $k = 1, 2, \dots$:

Indeed, it follows from the proof of Theorem 4.6 that this problem is equivalent to the equation

$$\eta = \mu\mathcal{B}\eta$$

(see (4.23)–(4.25)).

Nontrivial solutions of the Neumann–Newton problem (for the first series) and, seemingly, of the Dirichlet problem (for the second series) correspond to exceptional values of parameter λ from (4.41).

In particular, if

$$Lu + \lambda a u = 0, \quad \partial u = 0, \tag{4.47}$$

then we have, by virtue of formula (3.43),

$$u = -A^{-1}(\lambda a u), \quad u = A^{-1/2}\eta,$$

and then

$$\eta + \lambda\mathcal{A}\eta = 0, \quad \mathcal{A} = A^{-1/2} a A^{-1/2}.$$

Therefore, eigenvalues λ in this problem are equal to

$$\lambda = \lambda_k = -\lambda_k^{-1}(\mathcal{A}), \quad k = 1, 2, \dots;$$

they correspond to nontrivial solutions of problem (4.47).

To prove the statement for the second series of (4.41), we consider the Dirichlet problem of the form

$$Lu + \lambda au = 0, \quad \gamma u = 0.$$

4.3. Problems of S. G. Krein type. Problems of such type arise in studies of normal motion of viscous fluid in nonfull vessels (see [8, 38, 39] and [31, Sec. 7]).

Such problems can be generalized to a spectral problem, which can be expressed in terms of operators from the abstract Green formula (3.29) and a spectral parameter from the equation and the boundary condition.

The problem is to find nonzero elements $u \in V$ satisfying the following relations:

$$Lu = \lambda au, \quad \lambda\partial u = J\alpha\gamma u, \tag{4.48}$$

where λ is a spectral parameter, the operator $a \in \mathcal{L}(E)$, $a \gg 0$, and α and J are the same as above in Secs. 4.1 and 4.2.

Lemma 4.9. *Problem* (4.48) *has a nonzero eigenvalue of infinite order corresponding to the proper subspace*

$$M_0 := \big\{u = (u_1, \dots, u_q) \in M : \ \gamma_{jk1}u_j = \gamma_{kj1}u_k = 0, \\ \gamma_{jkl}u_j = \gamma_{kjl}u_k =: \varphi_{jkl} \in (G_+)_{jkl}, \ l = 2, 3, \ k > j; \ \ \gamma_{jj1}u_j = \gamma_{jj3}u_j = 0, \ j = \overline{1, q}\big\}.$$

Proof. From (4.48), we have for $\lambda = 0$

$$Lu = 0, \quad J\alpha\gamma u = 0. \tag{4.49}$$

Therefore, $u \in M$ and $\alpha\gamma u = 0$, since $J^2 = I$ and, consequently, $J = J^{-1}$. Similarly to Lemma 4.8, we take elements from $M \subset V$ that satisfy relations (4.33). □

Separating these trivial solutions and assuming that $\lambda \neq 0$, we obtain the following problem:

$$\eta = \lambda\mathcal{A}\eta + \lambda^{-1}\mathcal{B}\eta, \quad \eta = A^{1/2}u, \ \lambda \neq 0. \tag{4.50}$$

We call this problem the *problem of S. G. Krein type*. The operators $\mathcal{A}$ and $\mathcal{B}$ here are the same as in the Stefan problem (Sec. 4.1), i.e., they are defined by relations (4.12) and (4.13). The statements of Lemmas 4.1–4.4 hold.

Consider the simplest case of the problems of S. G. Krein type, that is, the case where the operator α in (4.48), given by (4.8), is replaced by the identity operator, i.e., $\alpha\gamma u = \gamma u$, and γu is defined according to (3.27). Then the number $\lambda = 0$ is not an eigenvalue of the problem:

$$Lu = \lambda au, \quad \lambda\partial u = J\gamma u.$$

Therefore, the following problem having only trivial solutions arises from (4.49) after substitution of α for I:

$$Lu = 0, \quad J\gamma u = 0.$$

Thus, we consider the following problem as $\lambda \neq 0$:

$$\eta = \lambda\mathcal{A}\eta + \lambda^{-1}\mathcal{B}\eta, \quad \eta = A^{1/2}u \in E, \ \mathcal{A} = A^{-1/2}aA^{-1/2}, \ \mathcal{B} = Q^*JQ. \tag{4.51}$$

Problems of type (4.51) (for meromorphic S. G. Krein sheaves) generated a widespread studies in the sixties of the last century and later. Therefore, we give only some results based on papers of S. G. Krein and his disciples, Markus and Matsaev (see [43–45]), and Kopachevsky and Azizov.

Definition 4.2. A basis $\{\psi_k\}_{k=1}^{\infty}$ of the Hilbert space E that can be obtained from the orthinormal basis $\{\varphi_k\}_{k=1}^{\infty}$ according to the law

$$\psi_k = C\varphi_k, \quad k = 1, 2, \dots, \quad C, C^{-1} \in \mathcal{L}(E), \tag{4.52}$$

is called a *basis equivalent to a normalized basis* or a *Riesz basis.*

Definition 4.3. We say that the Riesz basis $\{\psi_k\}_{k=1}^{\infty}$ is a *p-basis* if

$$C = I + T, \quad T \in \mathfrak{S}_p \ (0 < p < \infty)$$

in (4.52). If $p = 2$, this basis is called a Bary basis.

Lemma 4.10. *The kernel of the operator* $\mathcal{B}$ *from* (4.51) *is the set of elements*

$$\ker \mathcal{B} = \bigl\{\eta = A^{1/2}u : \ \gamma u = 0\bigr\} = A^{1/2}N =: E_0. \tag{4.53}$$

Proof. Let $\eta \in \ker \mathcal{B}$, i.e., $\mathcal{B}\eta = Q^*JQ\eta = 0$. Then for any $\zeta \in E$ we have

$$(\mathcal{B}\eta, \zeta)_E = (Q^*JQ\eta, \zeta)_E = (JQ\eta, Q\zeta)_G = 0.$$

Here $Q\zeta = \gamma A^{-1/2}\zeta$ covers all G_+ since $A^{-1/2}\zeta$ covers all V while ζ is changing in E. Since G_+ is dense in G, we have $JQ\eta = 0$; moreover, since J is invertible we have $Q\eta = 0$. Equation (4.53) follows from here. □

Since the operators $\mathcal{A}$ and $\mathcal{B}$ are self-adjoint and compact in (4.51), we see that the following operator sheaf corresponds to this problem:

$$\mathcal{I} - \lambda\mathcal{A} - \lambda^{-1}\mathcal{B}. \tag{4.54}$$

This operator sheaf is a self-adjoint holomorphic operator-valued function of a parameter λ at the whole complex plane $\mathbb{C}$ except for the points $\lambda = 0$ and $\lambda = \infty$. Such functions are called Fredholm sheaves (see, e.g., [31, p. 74]). Since the sheaf under consideration is invertible, for example, at any nonzero point of the conjugate axis, it is regular. This implies that the spectrum of problem (4.51) is discrete and consists of isolated eigenvalues of finite multiplicities with limit points at $\lambda = 0$ and $\lambda = \infty$.

Theorem 4.8. *For the sheaf* (4.54), *let the following conditions be satisfied*:

$$\dim E_1 := \dim(E \ominus E_0) = \infty, \quad 4\|\mathcal{A}\| \cdot \|\mathcal{B}\| < 1. \tag{4.55}$$

Introduce the following numbers:

$$r_\pm := \bigl(1 \pm \sqrt{1 - 4\|\mathcal{A}\| \cdot \|\mathcal{B}\|}\bigr)/2\|\mathcal{A}\|, \quad 0 < r_- < r_+ < \infty.$$

Then the following assertions hold.

1°. *Problem* (4.51) *has a discrete real spectrum with limit points at* $\lambda = 0$ *and* $\lambda = \infty$.

2°. *The branch* $\{\lambda_{0n}\}_{n=1}^{\infty}$ *of eigenvalues situated on the interval* $[0, r_-]$ *refers to the limit point* $\lambda = 0$ *for* $J = I$. *In the indefinite case, i.e., if* $J \neq I$ *and* $J \neq -I$, *the two branches* $\{\lambda_{0n}^+\}_{n=1}^{\infty}$ *and* $\{\lambda_{0n}^-\}_{n=1}^{\infty}$ *of eigenvalues situated at intervals* $[0, r_-]$ *and* $[-r_-, 0]$, *respectively, refer to this limit point. The system of eigenelements which forms the Riesz basis in* E_1 *after projecting to* $E_1 = E \ominus E_0$ *refers to the whole collection of eigenvalues.*

3°. *The branch* $\{\lambda_{\infty k}\}_{k=1}^{\infty}$ *of eigenvalues situated at the interval* $[r_+, \infty)$ *refers to the limit point* $\lambda = \infty$. *The corresponding system of eigenelements forms the Riesz basis in* E.

The proof of Theorem 4.8 can be found in [42, Theorem 30.2] and [23, 43].

We also assume that

$$\mathcal{A} \in \mathfrak{S}_{p_\mathcal{A}}, \ \mathcal{B} \in \mathfrak{S}_{p_\mathcal{B}}, \quad p_\mathcal{A}, \ p_\mathcal{B} > 0 \tag{4.56}$$

in problem (4.51).

Theorem 4.9. *If conditions* (4.56) *are valid, then the following assertions hold.*

1°. *The system of eigenelements corresponding to the eigenvalues lying on the interval* $[-r_-, r_-]$ *after projection to* E_1 *forms a p-basis in* E_1 *as* $p \geq p_0$, *where*

$$p_0^{-1} = (p_\mathcal{A})^{-1} + (p_\mathcal{B})^{-1}.$$

2°. *The system of eigenelements corresponding to the eigenvalues at the interval* $[r_+, \infty)$ *forms a p-basis in* E *for the same* p.

The proof of these statements can be found in [10, 22–24, 32].

As a rule, the operators of Dirichlet, Neumann, and Newton boundary-value problems have a power asymptotic of eigenvalues in problems of mathematical physics. In this case, the following statement holds

Theorem 4.10. *Assume that the following conditions are valid in problem* (4.51):

$$\lambda_k(\mathcal{A}) = c_{\mathcal{A}} k^{-1/\alpha}[1 + o(1)], \qquad k \to \infty,\ c_{\mathcal{A}} > 0,\ \alpha > 0;$$
$$\lambda_k^{\pm}(\mathcal{B}) = \pm c_{\mathcal{B}}^{\pm} k^{-1/\beta_\pm}[1 + o(1)], \quad k \to \infty,\ c_{\mathcal{B}}^{\pm} > 0,\ \beta_\pm > 0.$$

Then the following assertions hold.

1°. *For the branch* $\{\lambda_{\infty k}\}_{k=1}^{\infty} \subset [r_+, \infty)$ *of eigenvalues with the limit point* $+\infty$, *the following asymptotic formula holds*:

$$\lambda_{\infty k} = (\lambda_k(\mathcal{A}))^{-1}[1 + o(1)], \quad k \to \infty.$$

2°. *For the branches* $\{\lambda_{0k}^{+}\}_{k=1}^{\infty} \subset [0, r_-]$ *and* $\{\lambda_{0k}^{-}\}_{k=1}^{\infty} \subset [-r_-, 0]$ *of eigenvalues with limit points* ± 0, *the following asymptotic formula holds*:

$$\lambda_{0k}^{\pm} = \lambda_k^{\pm}(\mathcal{B})[1 + o(1)], \quad k \to \infty.$$

This theorem directly follows from papers [44, 45].

Thus, the asymptotic behavior of three (or two) branches of eigenvalues in problem (4.51) is defined by the corresponding asymptotic behavior of branches of eigenvalues of operators $\mathcal{A}$ and $\mathcal{B}$. In particular, there is a branch of eigenvalues with a limit point at zero at negative semiaxis for $J \neq I$. This situation arises, for example, in the problem of instability for normal motion of a heavy viscous revolving fluid (see [31, p. 312–325].)

If the condition $4\|\mathcal{A}\| \cdot \|\mathcal{B}\| < 1$ is not valid (see (4.55)), then problem (4.51) can have, in addition, a finite number of complex conjugate pair of eigenvalues of finite multiplicities. In particular, in the hydrodynamical S. G. Krein problem, this condition is valid for the sufficiently large viscosity of the liquid and is not valid for sufficiently small viscosity. In the second case, Theorems 4.8–4.10 hold but in the corresponding subspaces of finite codimension.

4.4. Problems on the spectrum of bounded operators. The problem under consideration arose in the research of Yudovich on convection theory. He studied problems of stability of new normal motions of dynamical systems arising after bifurcation. This problem can be formulated in the following form in terms of operators connected with Green formula (3.29) (see [21].)

Let $u \in V$ and $\varphi \in G$ be unknown elements. We find nonzero solutions of the system of equations

$$Au + w = \lambda u,\ \gamma u = \lambda\varphi, \quad Lw = 0,\ \partial w = \varphi, \tag{4.57}$$

where A is an operator of a Hilbert pair $(V; E)$ and λ is a spectral parameter.

Let us transform problem (4.57) and reduce it to a spectral problem for a bounded self-adjoint operator acting in the space $E \oplus G$.

The last two equations from (4.57) and the second auxiliary problem (Sec. 3.4) imply that

$$w = T_M\varphi,$$

and then we have the following problem instead of (4.57):

$$Au + T_M\varphi = \lambda u, \quad \gamma u = \lambda\varphi. \tag{4.58}$$

Since $T_M\varphi \in M \subset V$ as $\varphi \in G$, it follows from the first equation that $Au \in V = \mathcal{D}(A^{1/2})$. Therefore, we apply operator $A^{1/2}$ from the left in the first equation of (4.57) and substitute $\eta = A^{1/2}u \in \mathcal{D}(A) \subset E$:

$$\begin{aligned} A\eta + Q^*\varphi = \lambda\eta, &\quad \eta \in E,\ Q^* = A^{1/2}T_M, \\ Q\eta = \lambda\varphi, &\quad \varphi \in G,\ Q = \gamma A^{-1/2}. \end{aligned} \tag{4.59}$$

Introduce the operator

$$\mathfrak{A}_0 := \mathrm{diag}(A; 0), \quad \mathcal{D}(\mathfrak{A}_0) = \mathcal{D}(A) \oplus G, \tag{4.60}$$

which is considered unperturbed; then we can consider problem (4.59) as a problem about the spectrum of a perturbed (bounded by terms outside the diagonal) self-adjoint operator

$$\mathfrak{A} := \begin{pmatrix} A & Q^* \\ Q & 0 \end{pmatrix} : \mathcal{D}(A) \oplus G \to E \oplus G. \tag{4.61}$$

Note that since the operator $\mathfrak{A}_0$ is self-adjoint and the operators Q and Q^* are bounded and compact (Lemma 4.2), we have that the operator $\mathfrak{A}$ is an unbounded self-adjoint operator. The problem is to find how the spectrum of the initial operator $\mathfrak{A}_0$ changes if this operator is perturbed by terms Q^* and Q outside the diagonal.

Let us consider this problem and make the properties of solutions of spectral problem (4.59) clear.

Lemma 4.11. *The number $\lambda = 0$ is not an eigenvalue of problem* (4.59)*, i.e.,* $\ker \mathfrak{A} = \{0\}$.

Proof. For $\lambda = 0$, we have from (4.59)

$$(A\eta, \eta)_E + (Q^*\varphi, \eta)_E = 0, \quad (\varphi, Q\eta)_G = 0.$$

It follows from here that $(A\eta, \eta) = \|\eta\|_V^2 = 0$ and $\eta = 0$. Since the operator $Q^* = A^{1/2}T_M$ is invertible, we obtain from the relation $Q^*\varphi = 0$ that $\varphi = 0$. □

Lemma 4.12. *Problem* (4.59) *is equivalent to the following spectral problem for a meromorphic spectral sheaf*:

$$\xi = \lambda A^{-1}\xi - \lambda^{-1}B\xi, \quad B := A^{-1/2}Q^*QA^{-1/2}, \quad \xi = A^{1/2}\eta. \tag{4.62}$$

Proof. 1°. Since for solutions of problem (4.59), we have $\lambda \neq 0$, we see that $\varphi = \lambda^{-1}Q\eta$, and then

$$A\eta + \lambda^{-1}Q^*Q\eta = \lambda\eta, \quad \eta \in \mathcal{D}(A) \subset E. \tag{4.63}$$

Substituting $\eta = A^{-1/2}\xi$ and applying operator $A^{-1/2}$ from the left, we obtain Eq. (4.62).

2°. Conversely, if Eq. (4.62) satisfies with operator $B = A^{-1/2}Q^*QA^{-1/2}$, then we obtain the following equation after a reverse substitution $\xi = A^{1/2}\eta$:

$$A^{1/2}\eta = \lambda A^{-1/2}\eta - \lambda^{-1}A^{-1/2}Q^*Q\eta. \tag{4.64}$$

It follows from this equation that $\eta \in \mathcal{D}(A)$. Therefore, applying operator $A^{1/2}$ from the left, we obtain Eq. (4.63), and introducing φ, we obtain problem (4.59). □

The operators A^{-1} and B in problem (4.62) are compact and self-adjoint. Hence $A^{-1} > 0$ and $B \geq 0$. Therefore, the self-adjoint Fredholm sheaf $I - \lambda A^{-1} + \lambda^{-1}B$, which is invertible at any point of the complex plane outside the real axis, refers to problem (4.62). This follows from the fact that problem (4.59), which is equivalent to problem (4.62), can have a spectrum only at the real axis. It shows that the spectrum of problems (4.62) and (4.59) can be discrete with possible limit points $\lambda = \infty$ and $\lambda = 0$.

Theorem 4.11. *Problem* (4.62) *and the equivalent problem* (4.59) *have a real spectrum consisting of two branches of eigenvalues $\{\lambda_k^-\}_{k=1}^\infty \subset [-\|B\|, 0]$ and $\{\lambda_k^+\}_{k=1}^\infty \subset [\lambda_1(A), +\infty)$ of the finite multiplicity with limit points -0 and $+\infty$ respectively.*

The following estimates hold for eigenvalues λ_k^-:

$$-\lambda_k(B) \leq \lambda_k^- \leq -\lambda_k(B)/(1 + \lambda_k(B)\|A^{-1}\|), \quad k = 1, 2, \ldots, \tag{4.65}$$

and for the eigenvalues λ_k^+, the following estimates are valid:

$$\lambda_k(A) \leq \lambda_k^+ \leq \lambda_k(A) + \|B\|, \quad k = 1, 2, \ldots. \tag{4.66}$$

The eigenelements of problem (4.59) *form an orthogonal basis in space $E \oplus G$.*

Proof. Since the operator $\mathfrak{A}$ from (4.61) is a compact perturbation of operator $\mathfrak{A}_0$ (see (4.60)) and the operator $\mathfrak{A}_0$ has the points $+\infty$ and 0 as the limit spectrum, we have, by virtue of the well-known Weyl theorem, that the operator $\mathfrak{A}$ has the same points as the limit spectrum. Since $\lambda = 0$ is not an eigenvalue of problem (4.59) (see Lemma 4.11), we have that there exists a branch $\{\lambda_k^-\}_{k=1}^{\infty}$ of real eigenvalues of finite multiplicities with a limit point at zero. The branch $\{\lambda_k^+\}_{k=1}^{\infty}$, $\lim\limits_{k\to\infty} \lambda_k^+ = +\infty$ refers to the limit point $\lambda = +\infty$.

Let λ be an eigenvalue of problem (4.62) and $\xi \neq 0$ be the corresponding eigenelement. Then

$$\lambda^2 (A^{-1}\xi, \xi)_E - \lambda(\xi, \xi)_E - (B\xi, \xi)_E = 0. \tag{4.67}$$

It follows from here that eigenvalues λ can be found among the values of functionals

$$p_{\pm}(\xi) := \frac{(\xi,\xi)_E \pm \sqrt{(\xi,\xi)_E^2 + 4(A^{-1}\xi,\xi)_E \cdot (B\xi,\xi)_E}}{2(A^{-1}\xi,\xi)_E}, \quad \xi \neq 0,$$

that are roots of Eq. (4.67). They are called the Rayleigh functionals (see [1] and [31, p. 81]). For $p_+(\xi)$, we have

$$p_+(\xi) \geq \frac{(\xi,\xi)_E}{(A^{-1}\xi,\xi)_E} \geq \lambda_1(A) > 0,$$

and for $p_-(\xi)$, we have that

$$-\|B\| \leq p_-(\xi) \leq 0.$$

Therefore, $\{\lambda_k^-\}_{k=1}^{\infty} \subset [-\|B\|, 0]$ and $\{\lambda_k^+\}_{k=1}^{\infty} \subset [\lambda_1(A), +\infty)$.

Let us obtain the estimates (4.65) and (4.66) for the two branches of eigenvalues.

Introduce the functional

$$\widetilde{p}_+(\xi) := -\frac{1}{p_-(\xi)} = -\frac{2(A^{-1}\xi,\xi)_E}{(\xi,\xi)_E - \sqrt{(\xi,\xi)_E^2 + 4(A^{-1}\xi,\xi)_E \cdot (B\xi,\xi)_E}}$$

$$= \frac{(\xi,\xi)_E + \sqrt{(\xi,\xi)_E^2 + 4(A^{-1}\xi,\xi)_E \cdot (B\xi,\xi)_E}}{2(B\xi,\xi)_E}.$$

Then

$$0 \leq \widetilde{p}_+(\xi) - \frac{(\xi,\xi)_E}{(B\xi,\xi)_E} = \ldots = \frac{2(A^{-1}\xi,\xi)_E}{(\xi,\xi)_E + \sqrt{(\xi,\xi)_E^2 + 4(A^{-1}\xi,\xi)_E \cdot (B\xi,\xi)_E}} \leq \frac{(A^{-1}\xi,\xi)_E}{(\xi,\xi)_E} \leq \|A^{-1}\|.$$

We obtain the following inequalities:

$$\frac{(\xi,\xi)_E}{(B\xi,\xi)_E} \leq \widetilde{p}_+(\xi) \leq \frac{(\xi,\xi)_E}{(B\xi,\xi)_E} + \|A^{-1}\|.$$

These inequalities and the variation principles for polynomial operator sheaves (see [1] and [31, p. 81]) imply that

$$\frac{1}{\lambda_k(B)} \leq -\frac{1}{\lambda_k^-} \leq \frac{1}{\lambda_k(B)} + \|A^{-1}\|, \quad k = 1, 2, \ldots.$$

Thus, we obtain inequalities (4.65). Inequalities (4.66) are proved similarly.

The fact that problem (4.59) has a discrete spectrum with a finite number (two) of limit points (see, e.g., [51, Secs. 146–150] and [9, p. 268]) implies that eigenelements of problem (4.59) form an orthogonal basis. □

Theorem 4.11 and inequalities (4.65) and (4.66) imply that the following asymptotic formulas hold for the two branches of eigenvalues of problem (4.59):

$$\lambda_k^- = -\lambda_k(B)[1 + o(1)], \ k \to \infty; \quad \lambda_k^+ = \lambda_k(A) + O(1), \ k \to \infty.$$

Thus, passing from the operator $\mathfrak{A}_0$ to the operator $\mathfrak{A}$, the eigenvalues λ_k^+ are shifted to the right of the numbers $\lambda_k(A)$ no farther than $\|B\|$. Moreover, the second branch of eigenvalues $\{\lambda_k^-\}_{k=1}^{\infty} \subset [-\|B\|, 0]$ with a limit point at zero arises. Unstable modes of normal motions refer to such branch in convection problems.

4.5. Problems of normal motions of dynamical systems with surface dissipation of energy. Consider the following boundary-value problem for the wave equation in an arbitrary domain $\Omega \in \mathbb{R}^m$ with a Lipschitz boundary $\Gamma := \partial\Omega$:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f(t,x), \quad x \in \Omega, \tag{4.68}$$

with the boundary condition

$$\frac{\partial u}{\partial n} + u + \beta\frac{\partial u}{\partial t} = 0, \quad x \in \Gamma,\ \beta > 0, \tag{4.69}$$

and the initial conditions

$$u(0,x) = u^0(x),\ \frac{\partial u}{\partial t}(0,x) = u^1(x), \quad x \in \Omega. \tag{4.70}$$

Problems of this type are called problems with surface dissipation of energy (see [6, 7, 14, 15, 26]). The boundary condition (4.69) here contains the derivative with respect to t of the unknown function. Therefore, such conditions are called dynamical. The corresponding term $\beta(\partial u/\partial t)_\Gamma$, $\beta > 0$, generates surface dissipation of the total energy of the dynamical system.

Consider a homogeneous problem (4.68)–(4.70) without initial conditions and find its solutions in the form of normal motions:

$$u(t,x) = e^{-\lambda t}u(x), \quad x \in \Omega,\ \lambda \in \mathbb{C}.$$

The following spectral problem for the amplitude function $u(x)$ arises:

$$-\Delta u + \lambda^2 u = 0,\ x \in \Omega; \quad \frac{\partial u}{\partial n} + u - \lambda\beta u = 0,\ x \in \Gamma. \tag{4.71}$$

Problem (4.71) can be generalized to a multicomponent spectral conjugation problem with surface dissipation (or pumping) of energy which can be formulated in terms of operators from abstract Green formula (3.29) and operators of auxiliary boundary-value problems. The problem is to find nonzero elements $u \in V$ that are solutions of the following problem:

$$Lu + \lambda^2 au = 0, \quad \partial u - \beta\lambda(J\alpha)\gamma u = 0,\ \beta > 0,\ \lambda \in \mathbb{C}, \tag{4.72}$$

where L, ∂, and γ are the operators from Green formula (3.29), J and α are defined in (4.8) and (4.9) and below, and $a \in \mathcal{L}(E),\ a \gg 0$.

Using Theorem 3.3 on the representation of any element from V, we obtain from (4.72) that

$$u = A^{-1}(-\lambda^2 au) + T_M(\beta\lambda(J\alpha)\gamma u) = -\lambda^2 A^{-1}au + \beta\lambda T_M(J\alpha)\gamma u.$$

Finding u in the form $u = A^{-1/2}\eta,\ \eta \in E$, we obtain the equation

$$\mathcal{M}(\lambda)\eta := (\mathcal{I} - \beta\lambda\mathcal{B} + \lambda^2\mathcal{A})\eta = 0, \quad \eta \in E, \tag{4.73}$$

where the operators $\mathcal{A}$ and $\mathcal{B}$ are defined by formulas (4.12) and (4.13):

$$\mathcal{A} = A^{-1/2}aA^{-1/2}, \quad \mathcal{B} = Q^*(J\alpha)Q, \quad Q = \gamma A^{-1/2},\ Q^* = A^{1/2}T_M. \tag{4.74}$$

The spectral problem (4.73) describes normal motions of multicomponent systems with surface dissipation of energy (if $J = I$) and with energy pump (if $J \neq I$). It will be studied in detail in another paper.

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V. I. Voititsky
Faculty of Mathematics and Informatics, Chair of Mathematical Analysis,
Taurida National V. I. Vernadsky University
E-mail: `victor.voytitsky@gmail.com`
Nikolay Dmitrievich Kopachevsky
Faculty of Mathematics and Informatics, Chair of Mathematical Analysis,
Taurida National V. I. Vernadsky University
E-mail: `kopachevsky@list.ru`
Pavel Aleksandrovich Starkov
Faculty of Mathematics and Informatics, Chair of Mathematical Analysis,
Taurida National V. I. Vernadsky University
E-mail: `PavelStarkov@list.ru`