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ON LINEAR PROBLEMS WITH SURFACE DISSIPATION OF ENERGY

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Abstract. The first part of this work is devoted to applications of functional analysis methods to a linear initial-boundary value problem of mathematical physics with a surface dissipation of the energy. Its abstract analog is studied as well. The abstract Green formula for a triple of Hilbert spaces is used.

In the second part, spectral problems generated by linear initial-boundary value problems with a surface dissipation of the energy are studied. First we formulate the spectral problem of mathematical physics and the corresponding abstract problem. Further, we consider basic properties of the spectrum and show that it is rather specific in the case of considered problems; particular examples (one-dimensional and two-dimensional ones as well as an example of a cylindrical domain) are used for that. It turns out that the spectrum migrates in the complex plane, while the dissipation parameter changes from zero to infinity. Examples of numerical computations of the spectrum by means of the iteration method are provided. Further, we investigate the general setting of the spectral problem. Using a general result of Azizov, we prove that the spectrum of the generic problem is discrete and has a limiting point at infinity.

CONTENTS

Introduction

Evolution of dynamical systems with a surface dissipation of the energy (as well as the problem of its investigation) became familiar to the authors from the lecture of Prof. Chueshov delivered at the Fifteenth Crimean Autumn Math. School-Symposium (Laspi–Batiliman, 2004). In [2–4], infinite-dimensional dissipative dynamical systems (including systems with a surface dissipation of the energy) are studied.

1. Posing of the Initial-Boundary Value Problem of Mathematical Physics with a Surface Dissipation of the Energy

Let a domain $\Omega \subset \mathbb{R}^m$ have a Lipschitz boundary $\Gamma := \partial\Omega$. Consider the following initial-boundary value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \triangle u = f(t,x), \quad x \in \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial n} + u + \alpha\frac{\partial u}{\partial t} = 0, \quad x \in \Gamma,\ \alpha > 0, \tag{1.2}$$

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$$u(0,x) = u^0(x), \qquad \frac{\partial u}{\partial t}(0,x) = u^1(x), \quad x \in \Omega. \tag{1.3}$$

Here $u(t,x)$ is the sought function, functions $f(t,x), u^0(x)$, and $u^1(x)$ are given, and $\partial/\partial n$ denotes the outer normal derivative.

Note that boundary-value condition (1.2) contains a derivative with respect to t. That is why this condition is called *dynamical*. The corresponding term $\alpha(\partial u/\partial t)_\Gamma$ generates a surface dissipation of the full energy of the dynamical system if $\alpha > 0$. Thus, if $\alpha > 0$, then problem (1.1)–(1.3) is no longer a classical hyperbolic problem; it turns out that it is a parabolic problem.

2. Abstract Initial-Boundary Value Problem with Surface Dissipation of Energy

2.1. On the abstract Green formula for a triple of Hilbert spaces. Let a domain $\Omega \subset \mathbb{R}^m$ have a sufficiently smooth boundary $\Gamma := \partial\Omega$. For any $\eta(x) \in C^1(\overline{\Omega})$ and any $u(x) \in C^2(\overline{\Omega})$, the following Green function for the Laplace operator $\triangle := \sum\limits_{k=1}^{m} \dfrac{\partial^2}{\partial x_k^2}$ holds:

$$\int\limits_{\Omega} \eta(u - \triangle u)\, d\Omega = \int\limits_{\Omega} (\eta u + \nabla\eta\nabla u)\, d\Omega - \int\limits_{\Gamma} \eta \frac{\partial u}{\partial n}\, d\Gamma, \tag{2.1}$$

where $\partial/\partial n$ is the derivative with respect to the outer normal to Γ.

Considering Hilbert spaces $L_2(\Omega), \mathcal{H}^1(\Omega)$, and $L_2(\Gamma)$ (with the corresponding scalar products), we can represent (2.1) as follows:

$$(\eta, u - \triangle u)_{L_2(\Omega)} = (\eta, u)_{\mathcal{H}^1(\Omega)} - \left(\gamma\eta, \frac{\partial u}{\partial n}\right)_{L_2(\Gamma)}, \quad \gamma\eta := \eta|_\Gamma. \tag{2.2}$$

This can be generalized: first, functions $u(x)$ can be less smooth, secondly, the boundary $\partial\Omega$ can be Lipschitz, and, thirdly, E, F, and G can be an arbitrary triple of Hilbert spaces, while γ can be an abstract trace operator. The idea of such an abstract formula belongs to S. G. Kreĭn: it first appeared in [11, pp. 46-47]). In [8–10], the corresponding assertion is established in the most general form. Another case of an abstract Green formula is given in [13, pp. 187–189]).

We provide the corresponding result below.

Theorem 2.1. *Let the following conditions be satisfied for a triple of Hilbert spaces*

$$\{E, (\cdot,\cdot)_E\}, \{F, (\cdot,\cdot)_F\}, \{G, (\cdot,\cdot)_G\} :$$

(1) *The space F is imbedded in E densely and compactly, i.e., F is dense in E and*

$$\|u\|_E \le a\|u\|_F, \quad \forall u \in F. \tag{2.3}$$

(2) *An operator γ (an abstract trace operator) is defined on the space F such that γ is a bounded operator from F to G, γ maps F onto a dense set $\mathcal{R}(\gamma) =: G_+$ of the space G, G_+ is compactly imbedded in G, and*

$$\|\gamma u\|_G \le b\|u\|_F, \quad \forall u \in F. \tag{2.4}$$

Then there exist operators $L : F \to F^*$ and $\partial : F \to G_- := (G_+)^*$ uniquely defined by E, F, G (with the introduced scalar products), and by the operator γ such that the following Green formula holds:

$$\langle \eta, Lu\rangle_E = (\eta, u)_F - \langle \gamma\eta, \partial u\rangle_G \qquad \forall \eta, u \in F. \tag{2.5}$$

Here $\langle \eta, \psi\rangle_E$ and $\langle \xi, \varphi\rangle_G$ denote the linear functionals constructed on the elements $\eta \in F$, $\psi \in F^*$, $\xi \in G_+$, and $\varphi \in G_-$, respectively. They are the extensions of the functionals $(\eta, \psi)_E$ and $(\xi, \varphi)_G$ (with the preservation of their continuity), corresponding to the passing from $\psi \in E$ to $\psi \in F^* \supset E$ and from $\varphi \in G$ to $\varphi \in (G_+)^*$, respectively.

For $E = L_2(\Omega)$, $F = \widetilde{\mathcal{H}}^1(\Omega)$, and $G = L_2(\Gamma)$, we deduce the following result from Theorem 2.1.

Theorem 2.2. *For the triple of Hilbert spaces* $L_2(\Omega), \widetilde{\mathcal{H}^1}(\Omega)$ *(see below), and* $L_2(\Gamma)$ *(with the corresponding scalar products) and the trace operator* $\gamma : \widetilde{\mathcal{H}^1}(\Omega) \to L_2(\Gamma)$ *defined by the rule*

$$\gamma u := u|_\Gamma, \quad u \in \widetilde{\mathcal{H}^1}(\Omega), \tag{2.6}$$

the following Green formula holds:

$$\langle \eta, -\triangle u\rangle_\Omega = (\eta, u)_{1,\Omega} - \left\langle \gamma\eta, \frac{\partial u}{\partial n} + \gamma u \right\rangle_\Gamma, \qquad \forall \eta, u \in \widetilde{\mathcal{H}^1}(\Omega). \tag{2.7}$$

Here $\widetilde{\mathcal{H}^1}(\Omega)$ is the space with the scalar product

$$(u, v)_{1,\Omega} = \int_\Omega \nabla u \overline{\nabla v}\, d\Omega + \int_\Gamma u \cdot \overline{v}\, d\Gamma. \tag{2.8}$$

It follows from imbedding theorems and trace theorems for domains with Lipschitz boundaries that the norm in $\widetilde{\mathcal{H}^1}(\Omega)$ is equivalent to the standard norm of the space $\mathcal{H}^1(\Omega)$, i.e., to the norm

$$\|u\|^2_{\mathcal{H}^1(\Omega)} := \int_\Omega \left(|\nabla u|^2 + |u|^2\right) d\Omega. \tag{2.9}$$

Theorem 2.2 directly follows from the introduced spaces, the trace operator given by (2.6), general Theorem 2.1, and relation (2.5).

2.2. Abstract initial-boundary value problem with surface dissipation of energy: formulation. Let E, F, G, and the operator $\gamma : F \to G$ satisfy the conditions of Theorem 2.1. Then there exist operators $L : F \to F^*$ and $\partial : F \to (G_+)^*$ such that (2.5) holds.

The abstract initial-boundary value problem is posed as follows: to find a function $u = u(t)$ valued in F and satisfying the hyperbolic equation

$$\frac{d^2u}{dt^2} + Lu = f(t) \quad (\text{in}\, E), \tag{2.10}$$

the boundary-value condition

$$\partial u + \alpha\frac{d}{dt}(\gamma u) = 0 \quad (\text{in}\, G), \quad \alpha > 0, \tag{2.11}$$

and the initial-value conditions

$$u(0) = u^0,\, u'(0) = u^1. \tag{2.12}$$

Here L, γ, and ∂ are the operators from (2.5).

If $E = L_2(\Omega)$, $F = \widetilde{\mathcal{H}^1}(\Omega)$, and $G = L_2(\Gamma)$, then

$$Lu = -\triangle u, \quad \partial u = \frac{\partial u}{\partial n} + \gamma u. \tag{2.13}$$

Thus, problem (1.1)–(1.3) is a particular case of abstract problem (2.10)–(2.12) for $E = L_2(\Omega)$, $F = \widetilde{\mathcal{H}^1}(\Omega)$, $G = L_2(\Gamma)$, and $\gamma u := u|_\Gamma$.

Applying the full energy balance law to the solutions of problem (2.10)–(2.12) such that all terms of (2.10) and (2.11) are continuous functions of t, we obtain the identity

$$\frac{d}{dt}\left\{\frac{1}{2}\left\|\frac{du}{dt}\right\|^2_E + \frac{1}{2}\|u(t)\|^2_F\right\} = -\alpha\left\|\frac{d}{dt}(\gamma u)\right\|^2_G + \left(f(t), \frac{du}{dt}\right)_E. \tag{2.14}$$

For $f(t) \equiv 0$, this implies that the full energy of the system i.e., the differentiated sum at the left-hand side of (2.14), decreases because of the dissipation phenomena taking place in the space G (the surface dissipation).

2.3. Auxiliary abstract boundary-value problems. Consider auxiliary abstract boundary-value problems. Examples of such problems based on the abstract Green formula are studied in [8–10].

The first auxiliary problem (the abstract analog of the Neumann problem for the Poisson equation): for a given element f, to find a solution v of the problem

$$Lv = f \quad (\text{in } E), \quad \partial v = 0 \quad (\text{in } G). \tag{2.15}$$

Definition 2.1. We say that an element $v \in F$ is a *weak solution* of problem (2.15) if the identity

$$(\eta, v)_F = \langle \eta, f \rangle_E \quad \forall \eta \in F \tag{2.16}$$

holds.

Lemma 2.1. *For any $f \in F^*$, there exists a unique weak solution $v = A^{-1}f$ of problem* (2.15). *For that weak solution, the first relation of* (2.15) *holds not in E, but in $F^* \supset E$. Condition* (2.15) *holds in $(G_+)^* \supset G$. The operator A is an operator of the Hilbert space pair $(F; E)$, $\mathcal{D}(A) = F$, and $\mathcal{R}(A) = F^*$. A restriction of the operator A such that $\mathcal{R}(A) = E$ is an unbounded positive definite operator and its domain is dense in $F \subset E$. We have $\mathcal{D}(A^{1/2}) = F$. For $A : F \to F^*$, the following identity is valid*:

$$\langle \eta, Av \rangle_E = (\eta, v)_F = \left(A^{1/2}\eta, A^{1/2}v\right)_E \quad \forall \eta, v \in F. \tag{2.17}$$

If F is compactly imbedded in E, then the operator $A : \mathcal{D}(A) \subset E \to E$ has a discrete spectrum $\{\lambda_k(A)\}_{k=1}^{\infty}$ consisting of positive eigenvalues of finite multiplicities with the limiting point $\lambda = +\infty$.

Proof. The proof is provided in [10] (see also [8]). □

Also, note that a scale of spaces E^α, $-\infty < \alpha < \infty$, corresponding to the operator A can be constructed such that $E^0 = E$, $E^{1/2} = F$, and $E^{-1/2} = F^*$.

Note that that if $f \in E$, then $v = A^{-1}f$ is called a *generalized solution* of problem (2.15). Then $\mathcal{R}(A) = E$ and the set $A^{-1}\mathcal{R}(A) = \mathcal{D}(A)$ is dense in $F = \mathcal{D}(A^{1/2})$.

The second auxiliary problem (the abstract analog of the Neumann problem for the Laplace equation): for a given element ψ, to find a solution of the problem

$$Lw = 0 \quad (\text{in } E), \qquad \partial w = \psi \quad (\text{in } G). \tag{2.18}$$

Definition 2.2. We say that an element $w \in F$ is a *weak solution* of (2.18) if the following identity holds:

$$(\eta, w)_F = \langle \gamma\eta, \psi \rangle_G \quad \forall \eta \in F. \tag{2.19}$$

Note that those definitions of weak solutions for the first and the second auxiliary problems follow directly from (2.5), (2.15), and (2.18).

Lemma 2.2. *For any $\psi \in (G_+)^*$, there exists a unique weak solution $w = V\psi \in F$. The operator V is bounded on $(G_+)^*$; its range M is a subspace of F such that $Lw = 0$ for any $w \in M$ (any $w \in M$ is called an L-harmonic element).*

Proof. The proof is based on the inequality

$$|\langle \gamma\eta, \psi \rangle_G| \leq \|\gamma\eta\|_{G_+} \|\psi\|_{(G_+)^*} \leq \left(b\,\|\psi\|_{(G_+)^*}\right) \|\eta\|_F$$

and the Riesz lemma on the general form of a linear functional in the Hilbert space F. □

Lemma 2.3. *Any $u \in F$ can be represented as a sum of solutions of problems* (2.15) *and* (2.18), *i.e., in the following form*:

$$u = v + w = A^{-1}f + V\psi, \quad f = Lu \in F^*, \quad \psi = \partial u \in (G_+)^*. \tag{2.20}$$

Proof. The proof is given in [8], see Theorem 2.2 and Remark 2.1 of the present paper. □

2.4. Passing to the Cauchy problem for second-order differential equations in Hilbert spaces. The mentioned passing from problem (2.10)–(2.12) is possible if the sought function $u(t)$ is treated as a function of the independent variable t valued in F; presentation (2.20) and the operators A and V of auxiliary abstract boundary-value problems (2.15) and (2.18) should be used.

We have

$$u = v + w = A^{-1}\widehat{f} + V\widehat{\psi}, \quad \widehat{f} = Lu = f(t) - \frac{d^2u}{dt^2}, \quad \widehat{\psi} = -\alpha\frac{d}{dt}(\gamma u).$$

This yields

$$A^{-1}\frac{d^2u}{dt^2} + \alpha V\frac{d}{dt}(\gamma u) + u = A^{-1}f(t) \quad (\text{in } F). \tag{2.21}$$

Then the substitution

$$u(t) = A^{-1/2}\eta(t) \tag{2.22}$$

and the formal application of the operator $A^{1/2}$ from the left in (2.21) yields the equation

$$A^{-1/2}\frac{d^2}{dt^2}\left(A^{-1/2}\eta\right) + \alpha Q^*\frac{d}{dt}(Q\eta) + \eta = A^{-1/2}f(t) \quad (\text{in } E), \tag{2.23}$$

$$Q := \gamma A^{-1/2} : E \to G_+, \quad Q^* := A^{1/2}V : (G_+)^* \to E, \tag{2.24}$$

and the initial-value conditions

$$\eta(0) = A^{1/2}u^0, \quad \eta'(0) = A^{1/2}u^1. \tag{2.25}$$

Thus, we have the abstract Cauchy problem for a linear complete (for $\alpha > 0$) second-order differential equation in the Hilbert space E. If $\alpha = 0$, it is reduced to the Cauchy problem for a hyperbolic equation.

Remark 2.1. One can prove that the operators Q and Q^* are mutually adjoint and bounded. If G_+ is compactly imbedded in G, then the operator $Q : E \to G$ is compact and the restriction $Q^*|G$ is compact as well. Therefore,

$$B := Q^*Q : E \to E$$

is a nonnegative (self-adjoint) operator. Then

$$\operatorname{Ker} B = \operatorname{Ker} Q =: E_0 = A^{1/2}N := \{\eta \in E : \eta = A^{1/2}u, u \in F, \gamma u = 0\} \tag{2.26}$$

and the operator $B|E_1$ is positive on the orthogonal complement

$$E_1 := E \ominus E_0, \quad \dim E_1 = \infty.$$

If G_+ is compactly imbedded in G, then $B|E_1$ is compact. Hence, its spectrum consists of positive eigenvalues $\{\lambda_k(B)\}_{k=1}^{\infty}$ of finite multiplicities with the limiting point at the origin.

Thus, the original problem (2.10)–(2.12) is reduced to the Cauchy problem (2.23)–(2.25), while the properties of the operator coefficients of Eq. (2.23) are described in Lemma 2.1 and Remark 2.1. The latter problem will be studied below.

2.5. Application of the constructing semigroup theory. Now we pass from problem (2.23)–(2.25) to the Cauchy problem for a linear first-order equation with an operator coefficient that is a generator of a contracting semigroup. This allows us to prove the well-posedness of problem (2.23)–(2.25) and, correspondingly, of problem (2.10)–(2.12).

Transform problem (2.23)–(2.25), reducing it to a system of two differential equations. Introduce the new sought function $\zeta(t)$ as follows:

$$-i\eta = \frac{d\zeta}{dt}, \quad \zeta(0) = 0. \tag{2.27}$$

Then differentiate it with respect to t:

$$\frac{d^2\zeta}{dt^2} + i\frac{d\eta}{dt} = 0, \quad \zeta'(0) = -i\eta^0 = -iA^{1/2}u^0. \tag{2.28}$$

Taking into account (2.27) and (2.28), we write (2.23) as the following vector-matrix system of equations:

$$\begin{pmatrix} A^{-1/2} & 0 \\ 0 & I \end{pmatrix} \frac{d^2}{dt^2}\left[\begin{pmatrix} A^{-1/2} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \eta \\ \zeta \end{pmatrix}\right] + \begin{pmatrix} \alpha B & iI \\ iI & 0 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} A^{-1/2}f \\ 0 \end{pmatrix}. \tag{2.29}$$

Initial-value conditions for Eq. (2.29) are

$$(\eta(0);\zeta(0))^t = \left(A^{1/2}u^0;0\right)^t, \qquad (\eta'(0);\zeta'(0))^t = \left(A^{1/2}u^1;-iA^{1/2}u^0\right)^t. \tag{2.30}$$

Introduce the following denotations in (2.29), (2.30):

$$y(t) := \left(\frac{d\eta}{dt};\frac{d\zeta}{dt}\right)^t, \; f_0(t) := (A^{-1/2}f;0)^t, \; \mathcal{A}^{-1/2} := \begin{pmatrix} A^{-1/2} & 0 \\ 0 & I \end{pmatrix}, \; \mathcal{B} := \begin{pmatrix} \alpha B & iI \\ iI & 0 \end{pmatrix}. \tag{2.31}$$

Then problem (2.29), (2.30) has the form

$$\mathcal{A}^{-1/2}\frac{d}{dt}\left(\mathcal{A}^{-1/2}y\right) + \mathcal{B}y = f_0(t), \quad y(0) = y^0 := (A^{1/2}u^1;-iA^{1/2}u^0)^t. \tag{2.32}$$

Change the sought function:

$$z(t) := \mathcal{A}^{-1/2}y(t). \tag{2.33}$$

Applying the operator $\mathcal{A}^{1/2}$ from the left in (2.32), we get the Cauchy problem

$$\frac{dz}{dt} = -\mathcal{A}^{1/2}\mathcal{B}\mathcal{A}^{1/2}z(t) + \mathcal{A}^{1/2}f_0(t), \quad z(0) = (u^1;-iA^{1/2}u^0), \quad \mathcal{A}^{1/2}f_0(t) = (f(t);0)^t. \tag{2.34}$$

Thus, problem (2.23)–(2.25), which is the Cauchy problem for a total second-order linear differential equation considered in the space E and not resolved with respect to the principal derivative, is reduced to the Cauchy problem in the space $E^2 := E\oplus E$ for a first-order equation with one operator coefficient, which is the operator matrix

$$\mathcal{A}_{\mathcal{B}} := \mathcal{A}^{1/2}\mathcal{B}\mathcal{A}^{1/2} = \begin{pmatrix} A^{1/2} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \alpha B & iI \\ iI & 0 \end{pmatrix}\begin{pmatrix} A^{1/2} & 0 \\ 0 & I \end{pmatrix}. \tag{2.35}$$

This allows us to apply the theory of contracting semigroup of operators (see, e.g., [6]) to problem (2.34).

First we note that the operator matrix $\mathcal{B}$ in (2.31) is a bounded operator with a bounded inverse because the operator $B = Q^*Q$ is bounded and

$$\mathcal{B}^{-1} = \begin{pmatrix} 0 & -iI \\ -iI & \alpha B \end{pmatrix}. \tag{2.36}$$

Hence, the operator $\mathcal{A}_{\mathcal{B}}$ in (2.35) has a bounded inverse

$$\mathcal{A}_{\mathcal{B}}^{-1} := \mathcal{A}^{-1/2}\mathcal{B}^{-1}\mathcal{A}^{-1/2} = \begin{pmatrix} A^{-1/2} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & -iI \\ -iI & \alpha B \end{pmatrix}\begin{pmatrix} A^{-1/2} & 0 \\ 0 & I \end{pmatrix}; \tag{2.37}$$

therefore, it is defined on the whole space E^2. This means that

$$\mathcal{D}(\mathcal{A}_{\mathcal{B}}) = \mathcal{R}(\mathcal{A}_{\mathcal{B}}^{-1}), \quad \mathcal{R}(\mathcal{A}_{\mathcal{B}}) = \mathcal{D}(\mathcal{A}_{\mathcal{B}}^{-1}) = E^2. \tag{2.38}$$

Moreover, the operator $\mathcal{A}_{\mathcal{B}}$ is accretive, i.e.,

$$\mathrm{Re}(\mathcal{A}_{\mathcal{B}}z,z)_{E^2} = \left(\alpha B(A^{1/2}z_1),(A^{1/2}z_1)\right)_E = \alpha\left\|QA^{1/2}z_1\right\|_E^2 \geq 0 \quad \forall z = (z_1;z_2)^t \in \mathcal{D}(\mathcal{A}_{\mathcal{B}}). \tag{2.39}$$

It follows from (2.38) and (2.39) that $-\mathcal{A}_{\mathcal{B}}$ is a maximal dissipative operator on domain (2.38). Thus, this operator is a generator of a contractive semigroup of operators $U(t) := \exp(-t\mathcal{A}_{\mathcal{B}})$ and the solution of problem (2.34) can be represented by that semigroup.

We need the following general fact.

Theorem 2.3 (Phillips). *Let the Cauchy problem*

$$\frac{du}{dt} = Au + f(t), \quad u(0) = u^0, \quad t \geq 0, \tag{2.40}$$

be posed for functions $u(t)$ valued in a Banach space E. Let the operator A be a generator of a strongly continuous semigroup (C_0-semigroup) of operators $U(t)$. Let the following conditions be satisfied:

$$u^0 \in \mathcal{D}(A), \quad f(t) \in C^1([0,T];E). \tag{2.41}$$

Then problem (2.40) *has a unique strong solution $u(t)$ on $[0,T]$, which is represented as follows*:

$$u(t) = U(t)u^0 + \int_0^t U(t-s)f(s)ds. \tag{2.42}$$

Note that a function $u(t)$ is called a *strong solution* of problem (2.40) on $[0,T]$ if all terms of Eq. (2.40) are continuous functions of t for $t \in [0,T]$.

Using the Phillips theorem and conditions (2.41), we get conditions providing the existence of a strong solution of problem (2.34). From the second condition of (2.41), we obtain the following requirement for the right-hand side of Eq. (2.10):

$$f(t) \in C^1([0,T];E). \tag{2.43}$$

Further, the condition

$$z(0) = (u^1; -iA^{1/2}u^0)^t \in \mathcal{D}(\mathcal{A}^{1/2}\mathcal{B}\mathcal{A}^{1/2})$$

leads to the following relations:

$$A^{1/2}u^1 \in E, \quad \alpha BA^{1/2}u^1 + A^{1/2}u^0 \in \mathcal{D}(A^{1/2}) = F.$$

Theorem 2.4. *Let the following conditions be satisfied*:

(1) $f(t) \in C^1([0,T];E)$,
(2) $u^1 \in \mathcal{D}(A^{1/2}) = F$,
(3) $\alpha V\gamma u^1 + u^0 \in \mathcal{D}(A)$.

Then problem (2.34) *has a unique strong solution $z(t)$ on the segment* $[0,T]$.

Remark 2.2. Condition 3 of Theorem 2.4 means that the initial-value data in problem (2.34) should be consistent to ensure the strong solution of that problem. Using the representation $u^0 = v^0 + w^0$ (see Lemma 2.3) and noting that $w^0 = -\alpha V\gamma u^1$ due to the continuity as $t \to 0$, we see that condition 3 is reduced to the natural condition

$$u^0 - w^0 = v^0 \in \mathcal{D}(A). \tag{2.44}$$

2.6. Strong solvability theorem for abstract initial-boundary value problems. First we investigate the solvability of problem (2.23)–(2.25).

Definition 2.3. We say that problem (2.23)–(2.25) has a *strong solution* on the segment $[0,T]$ valued in $\mathcal{D}(A^{1/2}) = F$ if the term $A^{-1/2}\dfrac{d^2}{dt^2}(A^{-1/2}\eta)$ and the sum $\alpha Q^*\dfrac{d}{dt}(Q\eta)+\eta$ in Eq. (2.23) are continuous functions of $t \in [0,T]$ valued in F, Eq. (2.23) is satisfied for $t \in [0,T]$, and conditions (2.25) are satisfied.

From Theorem 2.3, we obtain the following result:

Theorem 2.5. *If conditions* 1–3 *of Theorem* 2.4 *are satisfied, then problem* (2.23)–(2.25) *has a unique strong solution on the segment* $[0,T]$.

Proof. Let conditions 1–3 of Theorem 2.4 be satisfied. Then problem (2.34) has a unique strong solution $z(t)$ on the segment $[0,T]$. This means that the functions $z_1(t)$ and $z_2(t)$ satisfy the system of equations

$$\frac{dz_1}{dt} + A^{1/2}(\alpha B A^{1/2} z_1 + i z_2) = f(t), \quad \frac{dz_2}{dt} + i A^{1/2} z_1 = 0 \tag{2.45}$$

and the initial-value conditions

$$z_1(0) = u^1, \ z_2(0) = -iA^{1/2}u^0 \tag{2.46}$$

and all terms of Eqs. (2.45) are continuous functions of $t \in [0,T]$ valued E; in particular, we have

$$\begin{gathered} z_1(t) \in C([0,T]; \mathcal{D}(A^{1/2})) \cap (C^1[0,T]; E), \qquad z_2(t) \in C^1([0,T]; E), \\ \alpha B A^{1/2} z_1(t) + i z_2(t) \in C([0,T]; E). \end{gathered} \tag{2.47}$$

The inverse changes of variables in (2.45) (see (2.33), (2.31)), i.e.,

$$z_1(t) = A^{-1/2}\frac{d\eta}{dt}, \quad z_2(t) = \frac{d\zeta}{dt}, \tag{2.48}$$

yield the system of equations

$$\frac{d}{dt}\left(A^{-1/2}\frac{d\eta}{dt}\right) + A^{1/2}\left(\alpha B\frac{d\eta}{dt} + i\frac{d\zeta}{dt}\right) = f(t), \quad \frac{d^2\zeta}{dt^2} + i\frac{d\eta}{dt} = 0 \tag{2.49}$$

and initial-value conditions (see (2.25), (2.27), (2.28))

$$\eta(0) = A^{1/2}u^0, \ \zeta(0) = 0, \ \eta'(0) = A^{1/2}u^1, \ \zeta'(0) = -iA^{1/2}u^0. \tag{2.50}$$

Taking into account conditions for $\eta(0)$ and $\zeta'(0)$, from the second equation of system (2.49), we have

$$\frac{d\zeta}{dt} + i\eta(t) = 0, \qquad \zeta(t) \in C^2([0,T]; E), \ \eta(t) \in C^1([0,T]; E). \tag{2.51}$$

Substituting $d\zeta/dt$ in the first equation of system (2.49), we obtain that the Cauchy problem

$$\frac{d}{dt}\left(A^{-1/2}\frac{d\eta}{dt}\right) + A^{1/2}\left(\alpha B\frac{d\eta}{dt} + \eta(t)\right) = f(t), \ \eta(0) = A^{1/2}u^0, \ \eta'(0) = A^{1/2}u^1 \tag{2.52}$$

has a unique strong solution $\eta(t)$ and all terms of (2.52) belong to $C([0,T]; E)$. In particular, we have

$$\alpha B\frac{d\eta}{dt} + \eta(t) \in C([0,T]; \mathcal{D}(A^{1/2})), \tag{2.53}$$

while any term of this sum is, generally speaking, an element of $C([0,T], E)$.

Applying the (bounded) operator $A^{-1/2}$ from the left in (2.52), we obtain the equation

$$A^{-1/2}\frac{d}{dt}\left(A^{-1/2}\frac{d\eta}{dt}\right) + \left(\alpha B\frac{d\eta}{dt} + \eta\right) = A^{-1/2}f(t), \tag{2.54}$$

where all terms (including the sum inside the parentheses) belong to $C([0,T]; \mathcal{D}(A^{1/2})) = C([0,T]; F)$.

The fact that the solution $\eta(t)$ of Eq. (2.54) is a solution of problem (2.23)–(2.25) valued in F follows from the identities

$$\frac{d}{dt}\left(A^{-1/2}\frac{d\eta}{dt}\right) \equiv \frac{d^2}{dt^2}(A^{-1/2}\eta), \qquad Q\frac{d\eta}{dt} \equiv \frac{d}{dt}(Q\eta). \tag{2.55}$$

This completes the proof of the theorem. □

Using this result, we prove the main assertion related to the solvability of problem (2.10)–(2.12).

Definition 2.4. We say that a function $u(t)$ is a *strong solution* of problem (2.10)–(2.12) on the segment $[0,T]$ if

$$u(t) \in C^2([0,T];E) \cap C^1([0,T];F), \tag{2.56}$$

$u(t)$ satisfies Eq. (2.10) such that each of its terms belongs to $C([0,T];E)$, satisfies condition (2.11) such that each of its terms belongs to $C([0,T];G_+)$, and satisfies conditions (2.12).

Theorem 2.6. *If conditions* 1–3 *of Theorem* 2.4 *are satisfied, then problem* (2.10)–(2.12) *has a unique strong solution on the segment* $[0,T]$.

Proof. Let conditions 1–3 of Theorem 2.4 be satisfied. Then the assertions of Theorem 2.5 hold, i.e., problem (2.23)–(2.25) has a unique strong solution on the segment $[0,T]$ in the sense of Definition 2.3.

In (2.23)–(2.25), we use the inverse change of variables provided by (2.22): $u(t) = A^{-1/2}\eta(t)$. Then Eq. (2.23) takes the form

$$A^{-1/2}\frac{d^2u}{dt^2} + \left(\alpha A^{1/2} V \frac{d}{dt}(\gamma u) + A^{1/2}u\right) = A^{-1/2}f(t), \tag{2.57}$$

while conditions (2.25) take the form

$$u(0) = u^0, \ u'(0) = u^1. \tag{2.58}$$

It follows from the proof of Theorem 2.5 that all terms in Eq. (2.57) (including the term inside the parentheses) belong to $C([0,T];\mathcal{D}(A^{1/2}))$. Hence, $u(t) \in C^2([0,T];E)$. Since $\eta(t) \in C^1([0,T];E)$ (see (2.51)), it follows that $u(t) = A^{-1/2}\eta(t) \in C^1([0,T];F)$, i.e., (2.56) is satisfied for $u(t)$. It remains to check whether $u(t)$ satisfies Eq. (2.10) and condition (2.11) with the properties formulated at Definition 2.4.

Apply the (bounded) operator $A^{-1/2}$ to (2.57) from the left. This yields the equation

$$A^{-1}\frac{d^2u}{dt^2} + \left(\alpha V \frac{d}{dt}(\gamma u) + u\right) = A^{-1}f(t), \tag{2.59}$$

while all its terms (including the term inside the parentheses) belong to $C([0,T];\mathcal{D}(A))$. Introduce the following functions:

$$v(t) := A^{-1}\left(f(t) - \frac{d^2u}{dt^2}\right), \ w(t) := -\alpha V \frac{d}{dt}(\gamma u). \tag{2.60}$$

Then (2.59) implies that

$$u(t) = v(t) + w(t) \tag{2.61}$$

and

$$v(t) \in C([0,T];\mathcal{D}(A)), \ w(t) \in C([0,T];\mathcal{D}(A^{1/2})). \tag{2.62}$$

Then relations (2.60) imply that the functions $f(t)$, $v(t)$, and $w(t)$ satisfy (2.61) and we have the following relations:

$$\frac{d^2u}{dt^2} + Av = f(t), \quad \partial w + \alpha \frac{d}{dt}(\gamma u) = 0. \tag{2.63}$$

To obtain the former relation, the operator A is applied (this is possible by virtue of the first property from (2.62)), while the latter one holds due to the definition of the operator V in the second boundary-value problem (see (2.18)–(2.19)). Now we note that, by virtue of the definition of the operator A (see (2.15)–(2.17)), it follows from the first relation of (2.63) that

$$\frac{d^2u}{dt^2} + Lv = f(t), \quad \partial v = 0. \tag{2.64}$$

Further, the element w satisfies the relation

$$Lw = 0. \tag{2.65}$$

Obviously, any term of the first relation in (2.64) is an element of $C([0,T];E)$. Further, any term of the second relation in (2.63) is an element of $C([0,T];G_+)$. Indeed, since $u(t) \in C^1([0,T];F)$, we have $\gamma u \in C^1([0,T];G_+)$ (due to the definition of the operator γ). Hence, we have $d(\gamma u)/dt \in C([0,T];G_+)$.

Now we add the left-hand and right-hand sides of (2.64) and (2.65) as well as the second relations in (2.63) and (2.64). Then, taking into account (2.61), we conclude that the function $u(t)$ satisfies the equation

$$\frac{d^2u}{dt^2} + Lu = f(t), \tag{2.66}$$

where all terms belong to $C([0,T];E)$, and the boundary-value condition

$$\partial u + \alpha \frac{d}{dt}(\gamma u) = 0, \tag{2.67}$$

where all terms belong to $C([0,T];G_+)$. This completes the proof of the theorem. □

Using the established general fact, we formulate the final solvability result for the original initial-boundary value problem of mathematical physics with a surface dissipation of the energy (see (1.1)–(1.3)).

Theorem 2.7. *Let the following condition be satisfied for problem* (1.1)–(1.3) *considered in a domain* $\Omega \subset \mathbb{R}^m$ *with a Lipschitz boundary* Γ:

(1) $f(t,x) \in C^1([0,T];L_2(\Omega))$;
(2) $u^1(x) \in \widetilde{\mathcal{H}}^1(\Omega)$;
(3) $u^0(x) = v^0(x) + w^0(x)$, $v^0(x) \in \mathcal{D}(A) \subset \widetilde{\mathcal{H}}^1(\Omega)$, $w^0(x) = -\alpha V \gamma u^1 \in \widetilde{\mathcal{H}}^1_h(\Omega)$.

Then this problem has a unique strong solution on the segment $[0,T]$, *i.e., there exists a function*

$$u(t,x) \in C^2([0,T];\, L_2(\Omega)) \cap C^1([0,T];\widetilde{\mathcal{H}}^1(\Omega))$$

satisfying Eq. (1.1) *such that each term of Eq.* (1.1) *is an element of* $C([0,T];\, L_2(\Omega))$, *satisfying condition* (1.2) *such that each term in condition* (1.2) *is an element of* $C([0,T];\mathcal{H}^{1/2}(\Gamma))$, *and satisfying conditions* (1.3).

3. Spectral Problems Generated by Initial-Boundary Value Problems with a Surface Dissipation of the Energy

3.1. Spectral problems posing. Consider problem (1.1)–(1.2), assuming that it is homogeneous and has no initial-value conditions, i.e., consider the following problem:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \text{ (in } \Omega), \quad \frac{\partial u}{\partial n} + u + \alpha\frac{\partial u}{\partial t} = 0 \text{ (in } \Gamma). \tag{3.1}$$

Definition 3.1. Similarly to problems of mechanics and hydrodynamics, we say that a function of the kind

$$u(t,x) = e^{-\lambda t}u(x),\ x \in \Omega, \tag{3.2}$$

is a *normal motion* of dynamical system (3.1) if there exist $\lambda \in \mathbb{C}$ and $u(x) \neq 0$ such that $u(t,x)$ satisfies the equation and the boundary-value condition in (3.1). Then the function $u(x)$ is called the *amplitude function*, while λ is called the *complex decay decrement.*

Obviously, the following spectral problem (containing a spectral parameter λ in the equation and in the boundary-value condition) arise for the amplitude function $u(x)$ from (3.1):

$$\Delta u - \lambda^2 u = 0 \text{ (in } \Omega), \quad \frac{\partial u}{\partial n} + u - \alpha\lambda u = 0 \text{ (in } \Gamma). \tag{3.3}$$

An abstract posing of spectral problem (3.3), based on the abstract Green formula, is possible as well (see [8–10]):

$$Lu + \lambda^2 u = 0 \text{ (in } E), \quad \partial u - \alpha\lambda\gamma u = 0 \text{ (in } G). \tag{3.4}$$

Evolution problem (2.23)–(2.25) studied above generates the spectral problem

$$(\lambda^2 A^{-1} - \lambda\alpha B + I)\eta = 0, \qquad \eta = A^{1/2}u \in E, \tag{3.5}$$

where the operators A^{-1} and B have the following properties.

1. The operator A is an operator of the Hilbert pair of spaces $(F; E)$ such that $\mathcal{D}(A) = F$ and $\mathcal{R}(A) = F^*$. If the range of a restriction of the operator A is equal to E, then that restriction is an unbounded positive definite operator defined on a domain dense in $F \subset E$ and $\mathcal{D}(A^{1/2}) = F$. If F is compactly imbedded in E, then the operator $A : \mathcal{D}(A) \subset E \to E$ has a spectrum $\{\lambda_k(A)\}_{k=1}^{\infty}$ consisting of positive eigenvalues of finite multiplicities with the limiting point $\lambda = +\infty$.

2. The operator B is a nonnegative (self-adjoint) operator such that

$$\operatorname{Ker} B =: E_0 = A^{1/2}N := \{\eta \in E : \eta = A^{1/2}u, \, u \in F, \, \gamma u = 0\} \tag{3.6}$$

and the operator $B|E_1$ is positive on the orthogonal complement

$$E_1 := E \ominus E_0, \quad \dim E_1 = \infty.$$

If G_+ is compactly imbedded in G, then $B|E_1$ is compact. Hence, its spectrum consists of positive eigenvalues $\{\lambda_k(B)\}_{k=1}^{\infty}$ such that their multiplicities are finite, they have a limiting point at the origin, and their asymptotic behavior is as follows:

$$\lambda_k(B) = c_B k^{-1/(m-1)}[1 + o(1)] \quad (k \to \infty), \quad c_B > 0. \tag{3.7}$$

Finally, the following spectral problem corresponds to evolution problem (2.34):

$$\mathcal{A}_{\mathcal{B}} z := \mathcal{A}^{1/2}\mathcal{B}\mathcal{A}^{1/2} z = \lambda z, \tag{3.8}$$

$$z = \mathcal{A}^{-1/2} y = -\lambda \begin{pmatrix} A^{-1/2}\eta \\ \zeta \end{pmatrix} = \begin{pmatrix} -\lambda u \\ -iA^{1/2}u \end{pmatrix} \in \mathcal{D}(\mathcal{A}_{\mathcal{B}}) \subset E \oplus E. \tag{3.9}$$

Below, we investigate problems (3.3), (3.4), (3.5), and (3.8).

3.2. Simplest properties of solutions of spectral problems. Assume that the dissipation parameter α is nonnegative. In particular, it can possess the values $\alpha = 0$ and $\alpha = +\infty$. Consider the simplest spectral properties of solutions of problems (3.3)–(3.8).

1. The number $\lambda = 0$ is not an eigenvalue of the spectral problems.

2. All eigenvalues of the spectral problems are located in the right-hand complex half-plane symmetrically with respect to the real axis.

3. If F is compactly imbedded in E and G_+ is compactly imbedded in G, then only discrete spectrum of problem (3.5) is possible, i.e., it can consist only of eigenvalues $\{\lambda_k(\alpha)\}_{k=1}^{\infty}$ of finite multiplicities with the limiting point $\lambda = \infty$.

Indeed, if the specified imbedding conditions are satisfied, then the operators A^{-1} and B in (3.5) are compact. Therefore, $L_\alpha(\lambda)$ is a so-called Fredholm operator pencil (see [5, pp. 37–40]), i.e., its structure is $I + \Phi(\lambda)$, where $\Phi(\lambda)$ is compact for any λ, the operator $L_\alpha(0) = I$ has a bounded inverse, and $\Phi(\lambda) := \lambda^2 A^{-1} - \lambda\alpha B$ is an analytic operator-function at any finite point of $\mathbb{C}$. This and the Gohberg theorem (see [5, p. 39]) imply the sought assertion.

In particular, this assertion holds for problem (3.3) because $\widetilde{\mathcal{H}}^1(\Omega) = F$ is compactly imbedded in $E = L_2(\Omega)$ and $G_+ = \mathcal{H}^{1/2}(\Gamma)$ is compactly imbedded in $G = L_2(\Gamma)$.

4. If $\alpha = 0$, then the spectrum of the problem is located at the imaginary axis (the hyperbolic case). If F is compactly imbedded in E, then the spectrum consists of the following eigenvalues of finite multiplicities:

$$\{\lambda_k\}_{k=1}^{\infty}, \; \lambda_k = \pm i\lambda_k^{1/2}(A), \; k = 1, 2, \ldots; \tag{3.10}$$

here $\lambda_k(A)$ are eigenvalues of the operator A, corresponding to the following abstract Neumann problem:

$$Lu = \lambda u \text{ (in } E), \quad \partial u = 0 \text{ (in } G). \tag{3.11}$$

In particular, this is valid for the Newton problem

$$Au := -\Delta u = \lambda u \text{ (in } \Omega), \quad \frac{\partial u}{\partial n} + u = 0 \text{ (in } \Gamma), \tag{3.12}$$

where the asymptotic behavior of $\lambda_k(A)$ is as follows:

$$\lambda_k(A) = c_A k^{2/m}[1 + o(1)] \quad (k \to \infty), \quad c_A > 0. \tag{3.13}$$

5. Let α be formally equal to ∞, i.e., the limit problems

$$\Delta u - \lambda^2 u = 0 \text{ (in } \Omega), \quad u = 0 \text{ (in } \Gamma) \tag{3.14}$$

and

$$Lu + \lambda^2 u = 0 \text{ (in } E\,), \quad \gamma u = 0 \text{ (in } G) \tag{3.15}$$

are considered instead of (3.3) and (3.4), respectively.

Then, if the subspace $N = \operatorname{Ker}\gamma \subset F$ is compactly imbedded in E, then the spectrum of problem (3.15), which is the Dirichlet problem, is discrete and consists of eigenvalues of finite multiplicities:

$$\{\lambda_k^{\pm}\}_{k=1}^{\infty}, \qquad \lambda_k^{\pm} = \pm i\lambda_k^{1/2}(A_0), \ k = 1, 2, \ldots; \tag{3.16}$$

here A_0 is the operator of problem (3.15). In particular, those facts are valid for problem (3.14) since the corresponding imbeddings are valid.

6. The examples considered below show the following phenomenon (which can be proved in the general case as well). If α goes up from zero, then the eigenvalues $\lambda_k(\alpha)$ such that $\lambda_k(0)$ coincide with numbers (3.10) are translated into the right-hand complex half-plane along the direction orthogonal to the imaginary axis; $\lambda_k(\alpha)$ tend to their limit values (3.16) (along the direction orthogonal to the imaginary axis) as $\alpha \to +\infty$.

In particular, this implies the following conclusion: for any k, there exists a critical value $(\alpha_k)_* > 0$. If $0 < \alpha < (\alpha_k)_*$, then the eigenvalue $\lambda_k(\alpha)$ goes away from the imaginary axis as α increases. If $\alpha > (\alpha_k)_*$, then the eigenvalue $\lambda_k(\alpha)$ goes towards the imaginary axis as α increases and the limit of that eigenvalue as $\alpha \to +\infty$ is located at the imaginary axis. In the examples considered below, that critical value α_* is the same for all eigenvalues λ and all spectrum is the infinity point for $\alpha = \alpha_*$.

Thus, spectral problems described above are rather specific. Therefore, they should be studied thoroughly.

3.3. One-Dimensional spectral problem. Consider a one-dimensional spectral problem ($m = 1$) of kind (3.3):

$$u''(y) - \lambda^2 u(y) = 0, \ 0 < y < 1, \ u(0) = 0, \ u'(1) = \alpha\lambda u(1). \tag{3.17}$$

The boundary Γ consists of the points 0 and 1 in this case. The boundary-value condition of kind (3.3) is set not on the full boundary, but merely at the point 1. The Dirichlet condition is set at the point 0. Moreover, we take only the expression $(\partial u/\partial n)_\Gamma = u'(1)$ instead of $\partial u/\partial n + u$ on Γ (for simplicity). We will see that this does not effect general qualitative conclusions about the structure of the spectrum, but just simplifies the problem.

We get the following characteristic equation for solutions of problem (3.17):

$$\coth\lambda = \alpha, \ 0 \le \alpha < \infty. \tag{3.18}$$

It is easy to see that the latter equation has solutions of different kinds for $0 < \alpha < 1$ and for $\alpha > 1$. If $0 < \alpha < 1$, then we obtain the sequence

$$\lambda = \lambda_p^-(\alpha) := c_-(\alpha) + i\pi(p - 1/2), \ c_-(\alpha) := \frac{1}{2}\ln\frac{1+\alpha}{1-\alpha}, \ p = \pm 1, \pm 2, \ldots. \tag{3.19}$$

If $\alpha > 1$, then we have

$$\lambda = \lambda_p^+(\alpha) := c_+(\alpha) + i\pi p,\ c_+(\alpha) := \frac{1}{2}\ln\frac{\alpha+1}{\alpha-1},\ p = 0, \pm1, \pm2, \dots. \tag{3.20}$$

If $\alpha = 1$, then problem (3.17) has no finite eigenvalues.

It follows from (3.19) and (3.20) that for $\alpha \neq 1$, the spectrum of problem (3.17) is discrete and consists of simple eigenvalues: $\{\lambda_p^-(\alpha)\}_{p=-\infty}^{+\infty}$ for $0 < \alpha < 1$ and $\{\lambda_p^+(\alpha)\}_{p=-\infty}^{+\infty}$ for $\alpha > 1$. Those eigenvalues are located either at the line $\operatorname{Re}\lambda = c_-(\alpha)$ (in the former case) or at the line $\operatorname{Re}\lambda = c_+(\alpha)$ (in the latter case). If α is changed, then $\lambda_p^\pm(\alpha)$ go along the line parallel to the real axis and we have

$$\lambda_p^- \to \infty\ (\alpha \to 1-0), \quad \lambda_p^+(\alpha) \to \infty\ (\alpha \to 1+0). \tag{3.21}$$

If $0 < \alpha < 1$, then problem (3.17) has no real eigenvalues. If $\alpha > 1$, then it has one real eigenvalue $\lambda_p^+(\alpha)$; since it is real, it follows that it is positive. It goes to the left as α increases. It tends to zero as $\alpha \to +\infty$.

For $p \neq 0$, the eigenvalues $\lambda_0^+(\alpha)$ tend to $\lambda_p^+(+\infty) = i\pi p$ as $\alpha \to +\infty$. Those limit values are solutions of the limit spectral Dirichlet problem

$$u''(y) - \lambda^2 u(y) = 0, \quad 0 < y < 1, \quad u(0) = u(1) = 0. \tag{3.22}$$

Thus, this simple example confirms the spectrum properties mentioned above (see assertion 6 at Sec. 3.2); in this particular example, the critical value α_* is equal to 1.

3.4. Two-Dimensional problems in rectangular domains. Consider the following spectral problem:

$$\Delta u - \lambda^2 u = 0 \text{ (in } \Omega), \quad u = 0 \text{ (in } S), \quad \frac{\partial u}{\partial y} - \lambda\alpha u = 0 \text{ (in } \Gamma), \tag{3.23}$$

where

$$\Omega := \{(x, y) : 0 < x < \pi,\ 0 < y < 1\} \subset \mathbb{R}^2, \tag{3.24}$$

$$\Gamma := \{(x, 1) : 0 < x < \pi\}, \tag{3.25}$$

and $S = \partial\Omega \backslash \Gamma$.

Similarly to problem (3.17), the dynamical boundary-value condition related to the surface dissipation of the energy is posed not on $\partial\Omega$, but only on its part $\Gamma \subset \partial\Omega$. The Dirichlet condition is posed on $S = \partial\Omega\backslash\Gamma$. Moreover, the term $u|_\Gamma$, which does not affect the general qualitative conclusions about the spectrum structure of problems of kind (3.3), is omitted.

Using the shape of the domain Ω and the boundary-value condition on S, we can separate the variables in the problem. This leads to the following equation:

$$\coth\sqrt{\lambda^2 + k^2} = \frac{\alpha\lambda}{\sqrt{\lambda^2 + k^2}},\ k = 1, 2, \dots, \quad \operatorname{Re}\sqrt{\lambda^2 + k^2} \geq 0. \tag{3.26}$$

Note that the latter with $k = 0$ coincides (formally) with Eq. (3.18) of the one-dimensional spectral problem. Therefore, it is reasonable to expect that the properties of solutions are similar in the case where $k \neq 0$. Also, note that $\sqrt{\lambda^2 + k^2} \neq 0$ in the considered example (we would have $\sinh(\sqrt{\lambda^2 + k^2}) \neq 0$ and $\cosh\sqrt{\lambda^2 + k^2} = 0$ otherwise).

It is easy to see that if $|\lambda|$ is large, while k is fixed, then solutions of Eq. (3.26) should be close to solutions of Eq. (3.18) because

$$\coth\sqrt{\lambda^2 + k^2} - \frac{\alpha\lambda}{\sqrt{\lambda^2 + k^2}} \sim \coth\lambda - \alpha \qquad (|\lambda| \to \infty).$$

This means that it would be reasonable to apply numerical methods (in particular, the iteration method) to an equation equivalent to Eq. (3.26). To get that equivalent equation, which is convenient for the iteration method if $|\lambda|$ is large enough, we use the same line of reasoning as for one-dimensional problem (3.26).

Introduce the following functions:

$$\zeta_1(\lambda) := \sqrt{\lambda^2 + k^2} = \lambda\sqrt{1 + \frac{k^2}{\lambda^2}}, \qquad \zeta_2(\lambda) := \left(\sqrt{1 + \frac{k^2}{\lambda^2}}\right)^{-1}. \tag{3.27}$$

Then Eq. (3.26) can be represented as follows:

$$\coth \zeta_1(\lambda) = \alpha\zeta_2(\lambda). \tag{3.28}$$

We will look for solutions of this equation close (for large $|\lambda|$) to solutions of Eq. (3.17) in the first quarter of the plane, i.e., for

$$\mathrm{Re}\lambda \geq 0, \qquad \mathrm{Im}\lambda \geq 0. \tag{3.29}$$

First we consider the case where $\alpha > 1$. From (3.28), taking into account that $\mathrm{Re}(\alpha\zeta_2(\lambda) - 1) > 0$ for large $|\lambda|$, we see that

$$\zeta_1(\lambda) = \frac{1}{2}\ln_0 \frac{\alpha\zeta_2(\lambda) + 1}{\alpha\zeta_2(\lambda) - 1} + i\pi p, \ p = 0, 1, 2, \ldots, \tag{3.30}$$

where $\ln_0 z$ is the principal branch of the function $\ln z$. Taking into account (3.27), we obtain the equation

$$\lambda\sqrt{1 + k^2\lambda^{-2}} = \frac{1}{2}\ln_0 \frac{\alpha + \sqrt{1 + k^2\lambda^{-2}}}{\alpha - \sqrt{1 + k^2\lambda^{-2}}} + i\pi p, \ p = 0, 1, 2, \ldots. \tag{3.31}$$

This equation is equivalent to the following one:

$$\lambda = \varphi^+_{kp}(\lambda; \alpha) := \frac{1}{2}\ln_0 \frac{\alpha + \sqrt{1 + k^2\lambda^{-2}}}{\alpha - \sqrt{1 + k^2\lambda^{-2}}} + i\pi p - \frac{k^2}{\lambda(1 + \sqrt{1 + k^2\lambda^{-2}})}, \quad k = 1, 2, \ldots, \ p = 0, 1, 2, \ldots. \tag{3.32}$$

It is possible to apply the iteration method to this equation because the function $\varphi^+_{kp}(\lambda; \alpha)$ is a contractive mapping for large values of $|\lambda|$: computations show that its derivative is small. Computations performed by means of the mathematical package Maple 9 lead to the following conclusions.

1. If k and $\alpha > 1$ are fixed, then the spectrum of problem (3.32) is discrete and consists of eigenvalues $\{\lambda^+_{kp}(\alpha)\}_{p=1}^{\infty}$ with the limit point $\lambda = \infty$ as $p \to \infty$. All eigenvalues $\{\lambda^+_{kp}(\alpha)\}_{p=1}^{\infty}$ are located in the strip $0 < \mathrm{Re}\,\lambda < c_+(\alpha)$.
2. The numbers $\lambda^+_{kp}(\alpha)$ tend to the imaginary axis as $\alpha \to +\infty$ along orbits close to horizontal ones; their limits are located at the imaginary axis.
3. The following property holds as $p \to +\infty$:

$$\mathrm{Re}\,\lambda^+_{kp}(\alpha) - c_+(\alpha) \to 0.$$

The case where $0 < \alpha < 1$ is investigated in the same way. Computations lead to the following conclusions.

1. If k and $0 < \alpha < 1$ are fixed, then the spectrum of problem (3.32) is discrete and consists of eigenvalues $\{\lambda^-_{kp}(\alpha)\}_{p=1}^{\infty}$ with the limit point $\lambda = \infty$ as $p \to \infty$. All eigenvalues $\{\lambda^-_{kp}(\alpha)\}_{p=1}^{\infty}$ are located to the right of the line $\mathrm{Re}\,\lambda = c_-(\alpha)$.
2. The eigenvalues $\lambda^-_{kp}(\alpha)$ go from the imaginary axis along orbits close to horizontal ones as α goes up from zero; they are located at the imaginary axis for $\alpha = 0$ and they tend to infinity as $\alpha \to 1 - 0$.
3. The following property holds as $p \to +\infty$:

$$\mathrm{Re}\,\lambda^-_{kp}(\alpha) - c_\alpha \to 0.$$

Finally, we note that the critical value of the dissipation parameter of two-dimensional problem (3.24)–(3.23) is the same as in one-dimensional problem (3.17): $\alpha_* = 1$. If $\alpha = \alpha_*$, then the spectrum of the problem consists only of the point of infinity.

3.5. Cylindrical domains in multidimensional spaces. Let $\Omega = \Gamma \times (0, h)$ be a cylindrical domain in $\mathbb{R}^{m+1}$. Let $\Gamma \subset \mathbb{R}^m$, S, and Γ_h be its lateral, lower, and upper surfaces, respectively.

Consider the problem

$$\Delta u - \lambda^2 u = 0 \text{ (in } \Omega), \quad u = 0 \text{ (in } S), \quad \frac{\partial u}{\partial y} - \lambda \alpha u = 0 \text{ (in } \Gamma_h), \tag{3.33}$$

where $u = u(x, y)$, $x = (x_1, \ldots, x_m) \in \Gamma$, $y \in (0, h)$, and the surface dissipation takes place only at Γ_h.

A partial separation of variables leads to the characteristic equation

$$\coth(\sqrt{\lambda^2 + \mu_k} h) = \alpha \frac{\lambda}{\sqrt{\lambda^2 + \mu_k}}, \tag{3.34}$$

where μ_k are eigenvalues of an unbounded, self-adjoint, and positive definite operator A_0 acting in $L_2(\Gamma)$. The operator A_0 has a discrete spectrum $\{\mu_k\}_{k=1}^{\infty}$, $0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_k \leq \ldots$, $\mu_k \to +\infty$ $(k \to \infty)$. We see that Eq. (3.34) coincides with Eq. (3.26) for $h = 1$ (just k is replaced by μ_k). Since $\mu_k \to +\infty$ as $k \to \infty$, it follows that Eq. (3.34) has the same general properties as Eq. (3.26).

Thus, for cylindrical domains in $\mathbb{R}^{m+1}$, problem (3.33) has a discrete spectrum migrating in $\mathbb{C}$ from the left to the right, while α increases from 0 to 1 (from the right to the left, while α increases from 1 to ∞), and consisting of the point of infinity for $\alpha_* = 1$.

3.6. General theorem on discrete spectrum. We come back to problem (3.5). This is a general spectral problem, i.e., the problem on the spectrum of the following operator pencil:

$$L_\alpha(\lambda) := \lambda^2 A^{-1} - \lambda \alpha B + I. \tag{3.35}$$

Our aim is to prove that the spectrum of $L_\alpha(\lambda)$ is discrete and its limit point is ∞ in the case of general position for the parameter α. Certain preliminary considerations precede the proof of that general fact.

In (3.35), we change the spectral parameter using the relation $\lambda = \mu^{-1}$. Then, instead of $L_\alpha(\lambda)$, we consider the operator pencil

$$M_\alpha(\mu) := \mu^2 I - \mu \alpha B + A^{-1}; \tag{3.36}$$

it should have a discrete spectrum with the limit point $\mu = 0$ under that change.

First we note that the specified fact depends on properties of the operator coefficients in the pencil $M_\alpha(\mu)$. Indeed, let Z be a Volterra operator acting in an arbitrary separable Hilbert space E (following [5, p. 33], we say that Z is a *Volterra operator* if it is completely continuous and it has no eigenvalues different from zero). For the operator Z, we construct the following quadratic self-adjoint operator pencil:

$$M_Z(\mu) := \mu^2 I + \mu B + C, \qquad B = Z + Z^*, \qquad C = Z^* Z. \tag{3.37}$$

Then we have

$$M_Z(\mu) = (\mu I + Z^*)(\mu I + Z). \tag{3.38}$$

This implies that the pencil $M_Z(\mu)$ is invertible for any $\mu \neq 0$ because both factors are invertible if $\mu \neq 0$. Thus, $M_Z(\mu)$ has the only point of the spectrum: $\mu = 0$.

Also, note that the eigenvalue problem for $M_Z(\mu)$, i.e., the problem

$$(\mu^2 I + \mu B + C)\varphi = 0, \quad \varphi \in E, \tag{3.39}$$

is equivalent (for $\mu \neq 0$) to the following eigenvalue problem in the space $E^2 = E \oplus E$:

$$\begin{pmatrix} 0 & C^{1/2} \\ -C^{1/2} & -B \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \mu \begin{pmatrix} \psi \\ \varphi \end{pmatrix}. \tag{3.40}$$

The operator

$$\mathfrak{A} := \begin{pmatrix} 0 & C^{1/2} \\ -C^{1/2} & -B \end{pmatrix} \tag{3.41}$$

is called the *linearizer* of pencil (3.39). The considerations provided above show that the linearizer $\mathfrak{A}$ of pencil (3.39) is a Volterra operator because it is completely continuous and and its spectrum (as well as the spectrum of (3.39) itself) consists only of the point $\mu = 0$.

Further investigations of the spectral problem for $L_\alpha(\lambda)$ from (3.35) and $M_\alpha(\mu)$ from (3.36) are based on a general assertion proved by Azizov after a discussion about problem (3.39) with the authors of the present paper. We recall (see [5, pp. 46, 120]) that a compact (completely continuous) operator belongs to the class $\mathfrak{S}_p$ if its s-numbers, i.e., the eigenvalues of the operator $(A^*A)^{1/2}$, are summable with power p:

$$\sum_{j=1}^{\infty}(s_j(A))^p = \sum_{j=1}^{\infty}\left(\lambda_j(A^*A)^{1/2}\right)^p < \infty$$

(here p denotes the infimum of the powers such that the series converges). We will need the following Matsaev theorem (see [5, p. 267]) as well.

Theorem 3.1. *Let $p \in (1, \infty)$. Let A be a Volterra operator. Then its real part $A_{\mathcal{R}} := (A + A^*)/2$ belongs to $\mathfrak{S}_p$ if and only if its imaginary part $A_J := (A - A^*)/(2i)$ belongs to $\mathfrak{S}_p$.*

Let us prove the following abstract result.

Theorem 3.2 (Azizov)**.** *Consider an operator pencil*

$$L(\lambda) := \lambda^2 I + \lambda B + C^2,\ B = B^* \in \mathfrak{S}_\infty,\ C = C^* \in \mathfrak{S}_\infty. \tag{3.42}$$

If

$$B \in \mathfrak{S}_p,\ C \in \mathfrak{S}_q \backslash \mathfrak{S}_p\ (q > p > 1) \tag{3.43}$$

or

$$B \in \mathfrak{S}_p \backslash \mathfrak{S}_q,\ C = \mathfrak{S}_q \quad (1 < q < p), \tag{3.44}$$

then the operator pencil $L(\lambda)$ has a countable set of nonzero eigenvalues of finite multiplicities with the limit point at the origin.

Proof. 1. Consider the linearizer of pencil (3.42), i.e., the operator

$$\mathfrak{A} = \begin{pmatrix} 0 & C \\ -C & -B \end{pmatrix} = \operatorname{Re}\mathfrak{A} + i\operatorname{Im}\mathfrak{A}, \tag{3.45}$$

$$\operatorname{Re}\mathfrak{A} = \begin{pmatrix} 0 & 0 \\ 0 & -B \end{pmatrix}, \qquad \operatorname{Im}\mathfrak{A} = \begin{pmatrix} 0 & -iC \\ iC & 0 \end{pmatrix}. \tag{3.46}$$

Since $B \in \mathfrak{S}_p$ and $C \in \mathfrak{S}_q$ due to conditions (3.43)–(3.44), it follows that $\operatorname{Re}\mathfrak{A} \in \mathfrak{S}_p$ and $\operatorname{Im}\mathfrak{A} \in \mathfrak{S}_q$. It follows from the same conditions that

$$\operatorname{Im}\mathfrak{A} \in \mathfrak{S}_q \backslash \mathfrak{S}_p\ (q > p) \ \text{ or } \ \operatorname{Re}\mathfrak{A} \in \mathfrak{S}_p \backslash \mathfrak{S}_q\ (q < p).$$

Then, by virtue of Theorem 3.1, $\mathfrak{A}$ is not a Volterra operator (in both cases). Therefore, it has nonzero eigenvalues of finite multiplicities, i.e., normal eigenvalues (see [5, p. 23]).

2. Suppose that the set of nonzero eigenvalues of the operator $\mathfrak{A}$ is finite and consists of numbers $\lambda_1, \lambda_2, \ldots, \lambda_s$. Consider the linear span

$$\mathcal{L} := \operatorname{Lin}\{\mathcal{L}_{\lambda_j}(\mathfrak{A})\}_{j=1}^{s}, \tag{3.47}$$

where $\mathcal{L}_{\lambda_j}(\mathfrak{A})$ is the root subspace (consisting of eigenvectors and associated vectors) of the operator $\mathfrak{A}$ corresponding to the eigenvalue $\lambda_j \neq 0$. Since $\dim \mathcal{L}_{\lambda_j}(\mathfrak{A}) < \infty$, we have $\dim \mathcal{L} < \infty$. Let

$$E^{\,2} = \mathcal{L} \oplus \mathcal{L}^{\perp}. \tag{3.48}$$

Since $\mathcal{L}$ is an invariant subspace for $\mathfrak{A}$, it follows that the matrix representation of the operator $\mathfrak{A}$ in orthogonal expansion (3.48) has the following form:

$$\begin{pmatrix} \mathfrak{A}_0 & \mathfrak{A}_{01} \\ 0 & \mathfrak{A}_1 \end{pmatrix}. \tag{3.49}$$

Then

$$\operatorname{Re}\mathfrak{A}_1 \in \mathfrak{S}_p,\ \operatorname{Im}\mathfrak{A}_1 \in \mathfrak{S}_q \backslash \mathfrak{S}p \quad (q > p) \tag{3.50}$$

or

$$\operatorname{Re}\mathfrak{A}_1 \in \mathfrak{S}_p \backslash \mathfrak{S}_q,\ \operatorname{Im}\mathfrak{A}_1 \in \mathfrak{S}_q \quad (q < p). \tag{3.51}$$

Indeed, let P be the orthoprojector on $\mathcal{L}^\perp$. Then

$$\text{Re } \mathfrak{A}_1 = P(\text{Re } \mathfrak{A})|\mathcal{L}^\perp,\ \text{Im } \mathfrak{A}_1 = P(\text{Im } \mathfrak{A})|\mathcal{L}^\perp.$$

The representations

$$\text{Re } \mathfrak{A} = \begin{pmatrix} \text{Re } \mathfrak{A}_0 & \frac{1}{2}\mathfrak{A}_{01} \\ \frac{1}{2}\mathfrak{A}_{01}^* & \text{Re } \mathfrak{A}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \text{Re } \mathfrak{A}_1 \end{pmatrix} + \begin{pmatrix} \text{Re } \mathfrak{A}_0 & \frac{1}{2}\mathfrak{A}_{01} \\ \frac{1}{2}\mathfrak{A}_{01}^* & 0 \end{pmatrix},$$

$$\text{Im } \mathfrak{A} = \begin{pmatrix} \text{Im } \mathfrak{A}_0 & \frac{1}{2i}\mathfrak{A}_{01} \\ -\frac{1}{2i}\mathfrak{A}_{01}^* & \text{Im } \mathfrak{A}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \text{Im } \mathfrak{A}_1 \end{pmatrix} + \begin{pmatrix} \text{Im } \mathfrak{A}_0 & \frac{1}{2i}\mathfrak{A}_{01} \\ -\frac{1}{2i}\mathfrak{A}_{01}^* & 0 \end{pmatrix}$$

and the fact that the latter terms on the right-hand sides of those relations are finite-dimensional imply that

$$\text{Re } \mathfrak{A}_1 \in \mathfrak{S}_p,\ \text{Im } \mathfrak{A}_1 \in \mathfrak{S}_q,$$

and

$$\text{Im } \mathfrak{A}_1 \in \mathfrak{S}_q \backslash \mathfrak{S}_p \quad (q > p) \tag{3.52}$$

or

$$\text{Re } \mathfrak{A}_1 \in \mathfrak{S}_p \backslash \mathfrak{S}_q \quad (q < p). \tag{3.53}$$

3. Taking into account (3.52) and (3.53), we use Theorem 3.1 again. This yields that $\mathfrak{A}_1$ is not a Volterra operator. Then it has a nonzero normal eigenvalue λ_0.

If $\lambda_0 \neq \lambda_j,\ j = 1, \ldots, s$, then the eigenvalue problem for the operator $\mathfrak{A}$ (representable as matrix (3.49)), i.e., the problem

$$\begin{pmatrix} \mathfrak{A}_0 - \lambda_0\mathfrak{I} & \mathfrak{A}_{01} \\ 0 & \mathfrak{A}_1 - \lambda_0\mathfrak{I} \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.54}$$

has a nonzero solution. Indeed, it follows from (3.54) that

$$\varphi_1 \in \text{Ker}(\mathfrak{A}_1 - \lambda_0\mathfrak{I}),$$
$$\varphi_0 = -(\mathfrak{A}_0 - \lambda_0\mathfrak{I})^{-1}\mathfrak{A}_{01}\varphi_1.$$

This leads to a contradiction with the fact that $\{\lambda_1, \ldots, \lambda_s\}$ are all eigenvalues of the operator $\mathfrak{A}$.

If λ_0 coincides with one of the numbers $\lambda_1, \ldots, \lambda_s$, then λ_0 is an eigenvalue both for $\mathfrak{A}_1$ and $\mathfrak{A}_0$. Then, if $\varphi_1 \neq 0$ is the corresponding eigenvector of the operator $\mathfrak{A}_1$, i.e., $(\mathfrak{A}_1 - \lambda_0\mathfrak{I})\varphi_1 = 0$, then it is easy to check that

$$(\mathfrak{A} - \lambda_1\mathfrak{I})^{p_1+1}(\mathfrak{A} - \lambda_2\mathfrak{I})^{p_2+1} \ldots (\mathfrak{A} - \lambda_s\mathfrak{I})^{p_s+1} \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.55}$$

where p_j is the maximal length of the Jordan chain of an eigenvector and its associated vectors corresponding to the eigenvalue $\lambda_j,\ j = 1, 2, \ldots, s$. Hence,

$$\begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix} \in \text{Lin}\,\{\mathcal{L}_{\lambda_j}(\mathfrak{A})\}_{j=1}^s = \mathcal{L} \subset E^2. \tag{3.56}$$

On the other hand, the element $\begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix}$ was chosen to be orthogonal to that linear span. We arrive at a contradiction to the fact that $\varphi_1 \neq 0$.

This and the previous contradictions prove the assertion of the theorem, i.e., the operator pencil $L(\lambda)$ from (3.42) has a countable set of nonzero normal eigenvalues with the limit point at the origin, provided that condition (3.43) or condition (3.44) is satisfied. $\square$

Remark 3.1. In orthogonal decomposition (3.48), we have

$$\begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix} \in \mathcal{L}, \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix} \perp \mathcal{L}, \qquad (\mathfrak{A} - \lambda_0 \mathfrak{J}) \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} \mathfrak{A}_{01}\varphi_1 \\ 0 \end{pmatrix} \in \mathcal{L}. \tag{3.57}$$

This yields property (3.55).

Theorem 3.2 implies the following assertion.

Corollary 3.1. *If the conditions of Theorem* 3.2 *are satisfied, then the spectrum of the operator pencil*

$$M(\mu) := I + \mu B + \mu^2 C^2 \tag{3.58}$$

is discrete and has a limit point at infinity.

Indeed, we have

$$M(\mu) = \lambda^{-2}(\lambda^2 I + \lambda B + C^2) = \lambda^{-2} L(\lambda), \ \lambda = \mu^{-1}, \tag{3.59}$$

while the spectrum of $L(\lambda)$ is discrete and its limit point is the origin.

The following result describes the properties of the spectrum of pencil (3.35).

Theorem 3.3. *Let $L_\alpha(\lambda)$ be an operator pencil from* (3.35) *such that either $B \in \mathfrak{S}_p$, $A^{-1/2} \in \mathfrak{S}_q \backslash \mathfrak{S}_p$ $(q > p > 1)$ or $B \in \mathfrak{S}_p \backslash \mathfrak{S}_q$, $A^{-1/2} \in \mathfrak{S}_q$ $(1 < q < p)$. Then the spectrum of $L_\alpha(\lambda)$ is discrete and its limit point is ∞. In particular, if the eigenvalues $\lambda_j(A^{-1})$ and $\lambda_j(B)$ are such that*

$$\lambda_j(A^{-1}) = c_A j^{-\alpha}[1 + o(1)], \ \lambda_j(B) = c_B j^{-\beta}[1 + o(1)], \ 0 < \alpha, \ \beta < 1, \ j \to \infty, \tag{3.60}$$

where

$$\alpha \neq 2\beta, \tag{3.61}$$

then the spectrum of $L_\alpha(\lambda)$ is discrete.

Proof. The first assertion of the theorem follows from Corollary 3.1, where μ is replaced by λ, B is replaced by $-\alpha B$, and C^2 is replaced by A^{-1}, i.e., $C = A^{-1/2}$.

Now we note that asymptotic formulae of kind (3.60) are typical for operators of boundary-value problems of mathematical physics. In particular, it follows from (3.60) that $B \in \mathfrak{S}_p$ for $p > \beta^{-1}$, while $A^{-1} \in \mathfrak{S}_q$ for $q > \alpha^{-1}$. Therefore, $A^{-1/2} \in \mathfrak{S}_{2q}$. Hence, the conditions of the first part of the theorem are satisfied if and only if property (3.61) takes place. $\square$

This assertion implies the following result for problem (3.3).

Theorem 3.4. *Let Ω be a domain in $\mathbb{R}^m$ with piecewise-smooth boundary $\Gamma = \partial\Omega$ (with nonzero inner and outer angles), $m \geq 3$. Then the spectrum of problem* (3.3) *is discrete and it consists of eigenvalues of finite multiplicities with limit point at infinity.*

Proof. Since problem (3.3) is equivalent to the eigenvalue problem for the pencil $L_\alpha(\lambda)$ from (3.35), it suffices to prove that the conditions of Theorem 3.3 are satisfied.

It follows from (3.13) that eigenvalues $\lambda_j(A^{-1})$ of the operator A^{-1} are such that (3.60) holds, where $\alpha = 2/m$. Eigenvalues of the operator B are given by (3.7), where $\beta = 1/(m-1)$. This implies that $\alpha, \beta < 1$ and $2\beta \neq \alpha$ for $m \geq 3$. This completes the proof. $\square$

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