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PROBLEM ON SMALL MOTIONS AND NORMAL OSCILLATIONS OF CAPILLARY VISCOUS LIQUIDS IN ROTATING VESSELS

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Abstract. The new recent results of the author are applied to study the problem. We begin from the problem posing. Then we consider the problem as a system of operator equations in a Hilbert space. Further, the initial-boundary value problem is reduced to the Cauchy problem for the abstract parabolic equation; this allows us to prove the unique solvability theorem. Then we study normal oscillations of the hydraulic system under the assumption of static stability with respect to the linear approximation. We prove results about the spectrum of the problem and prove that the system of root functions (eigenfunctions and associated functions) form a basis. Also, we prove that if the static stability assumption is not satisfied, then the inversion of Lagrange's theorem on the stability is valid.

CONTENTS

1. Introduction

1.1. On the history of the problem. Problems on small motions of a fluid in a vessel under conditions close to weightlessness, when capillary (surface) forces should be taken into account, were studied in 1960s–1980s in many papers (see, e.g., [19–21]). Operator approach to problems of this kind is described in detail in [16] (see also [13, 14]).

This initial-boundary value problem is special because not only the (Navier–Stokes) equation itself, but the boundary-value (kinematic) condition contains the derivative of the solution with respect to the time variable. Moreover, the order of the differential operator at the equilibrium surface of the fluid (it is the Laplace–Beltrami operator) coincides with the order of the main equation because capillary forces act. Thus, the problem becomes more complicated. The corresponding problem about normal motions, i.e., about solutions depending on the time as $e^{-\lambda t}$, $\lambda \in \mathbb{C}$, leads to a non-self-adjoint spectral problem that has not been sufficiently investigated up to now.

Also, note that mainly so-called generalized (weakened) solutions were studied so far, while the completeness (instead of basisness) of the root function system was proved for the spectral problem. In this paper, we make a certain step towards the proof of the existence of strong solutions; also, we prove the basisness of the root function system in the Abel–Lidskii sense.

The following fact (the inversion of Lagrange's theorem on the stability) is well known for systems with finite numbers of degrees of freedom. Consider a dissipative system (a system with friction) and its small motions with respect to a stationary motion mode (e.g., a quiescent mode or a uniform rotation). If the potential energy matrix of this system (it is self-adjoint) has a negative eigenvalue,

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i.e., the system is statically unstable with respect to the linear approximation, then it is dynamically unstable as well, i.e., there exist small motions of that system with respect to that stationary motion mode such that those small motions are exponentially increasing functions of time.

It is frequently not easy to find sufficient unstability conditions in fluid mechanics problems, i.e., for systems with infinite numbers of degrees of freedom. Methods of the theory of linear operators in Hilbert spaces (instead of matrix theory) are natural for this area. Also, the problem might become harder if the spectrum of the potential energy operator is not discrete.

In this paper, the specified problem is studied for small motions of a capillary viscous rotating fluid occupying a part of a container. We provide a detailed proof of the inversion of Lagrange's theorem on the stability for small motions close to a quiescent mode or a uniform rotation about the vertical axis. Pioneering results in this direction are contained in [8–10, 28]. In [13, 14, Vol. 2, Sec. 9.3], a more detailed proof is provided, but it is still far from completeness. This (and numerous useful considerations of this problem with Prof. V. A. Solonnikov) is the motivation to write this paper with detailed explanations. The author is grateful to Prof. V. A. Solonnikov for the invitation to Ferrara University (Italy) in June of 2004.

Also note that if the fluid is not viscous, but ideal, then a statically unstable system might become stable because of Coriolis (gyroscopic) forces (this is called the gyroscopic stabilization effect). If there are no such forces, then the inversion of Lagrange's theorem on the stability holds for the ideal capillary fluid. This is proved, e.g., in [19, pp. 280–282], [20, pp. 306–308], [16, p. 166], [13, p. 208].

1.2. Methods and results. To investigate weather linear initial-boundary value problems of fluid dynamics, methods of functional analysis, theory of boundary-value problems and spectral problems, theory of linear differential equations in Hilbert spaces, and spectral theory of non-self-adjoint operators are natural.

In Sec. 2, we provide the complete mathematical posing of the problem on small motions of a capillary viscous fluid in a uniformly rotating partially filled vessel. Then necessary function spaces and their equipments are introduced.

Further, we formulate the theorem on the existence of the Green formula for a triple of Hilbert spaces satisfying certain restrictions and an existence theorem for an abstract trace operator. That formula generalizes the classical Green formula for the Laplace operator greatly. We apply it to reduce the original initial-boundary value problem to a Cauchy problem for a system of differential-operator equations (Sec. 2.5).

In Sec. 3, the solvability of the initial-boundary value problem is studied. We prove (Sec. 3.1) that the specified problem is reduced to the Cauchy problem for abstract parabolic equations in Hilbert spaces, provided that the condition of static stability with respect to the linear approximation holds (see (2.85)). This allows us to prove the existence and uniqueness of a strong solution of the Cauchy problem associated with the original initial-boundary value problem (Theorem 3.1). If the solution has additional smoothness properties with respect to t (see Definition 3.3), then solutions of the original initial-boundary value problem have additional smoothness properties as well (Theorem 3.2).

Further, in Sec. 4, we study normal oscillations of a hydraulic system under the condition of static stability with respect to the linear approximation. The spectral problem is posed and the Abel–Lidskii definition of the basisness for systems of elements is provided. Then we formulate an abstract theorem (Theorem 4.1) providing sufficient conditions of basisness (in the Abel–Lidskii sense) for a system of root elements of a non-self-adjoint operator with discrete spectrum; this theorem allows us to find the asymptotics of characteristic values of that operator. This yields (see Sec. 4.3) the main result (Theorem 4.2) about the spectrum of the problem on normal oscillations of the capillary viscous rotating fluid, the basisness (in the Abel–Lidskii sense) of a system of root elements, and the asymptotic behavior of eigenvalues of the (discrete) spectrum for the specified problem. In particular, we find that if the condition of static stability with respect to the linear approximation is satisfied (i.e., the potential energy of the system has a rough minimum at the relative equilibrium state), then all normal motions of the hydraulic system are asymptotically damping.

In Sec. 5, we consider the case where the condition of static stability with respect to the linear approximation is not satisfied and the potential energy operator has at least one negative eigenvalue. First (in Sec. 5.1) we consider the case where the fluid does not rotate. We start from certain auxiliary assertions (Theorems 5.1–5.2 and Lemmas 5.1–5.5). Then we prove the main assertions for the non-rotating fluid (Theorems 5.3 and 5.4). They imply the inversion of Lagrange's theorem on the stability (Theorem 5.5) for $\omega_0 = 0$. Then, in Sec. 5.2, we we consider the case where the fluid rotates ($\omega_0 \neq 0$) and prove the inversion of Lagrange's theorem on the stability (Theorem 5.6).

Finally, note that the method of continuation of the solution by a parameter is applied here twice: one parameter is the fluid viscosity, while the second parameter ε is introduced artificially (see proofs of Theorems 5.4–5.6).

2. Mathematical Posing and Reduction to a System of Operator Equations

2.1. Physical and mathematical posing. Let a viscous incompressible fluid of density ρ fill a part of an arbitrary container (vessel) $\Omega \subset \mathbb{R}^3$ and rotate together with it (without perturbations) with a constant angular velocity $\vec{\omega}_0 = \omega_0 \vec{e}_3$, where $\vec{e}_3$ is the unit vector of the rotation axis Ox_3. Assume that the coordinate system $Ox_1x_2x_3$ is rigidly connected with the vessel, while the outer stationary force field $\vec{F}_0$ is the gravitational one and it acts along the rotation axis, i.e., $\vec{F}_0 = -g\vec{e}_3$, $g > 0$.

The following distribution of the pressure $P_0(x)$ takes place in the fluid under the condition of relative equilibrium:

$$P_0(x) = -\rho g x_3 + \frac{1}{2}\,\rho\omega_0^2(x_1^2 + x_2^2) + c \quad (\text{in } \Omega), \qquad c \in \mathbb{R}, \tag{2.1}$$

i.e., it is defined by actions of gravitational and centrifugal forces. If the gravitational forces are sufficiently large, i.e., the fluid can be treated as heavy, then the following condition is satisfied along the equilibrium surface Γ:

$$(P_0)_\Gamma = p_a, \tag{2.2}$$

where p_a is the external constant pressure. This and (2.1) imply that Γ is the following paraboloid of revolution:

$$x_3 = \frac{1}{2g}\,\omega_0^2(x_1^2 + x_2^2) + \frac{c - p_a}{\rho g}. \tag{2.3}$$

Here the constant c is obtained from the following condition:

$$\int\limits_\Omega d\Omega = V \tag{2.4}$$

(the volume of the domain Ω occupied by the fluid in the case of the solid-state rotation is equal to a given value V).

If the gravitational field is weak and the rotation of the system is slow, then the fluid should be treated as capillary, i.e., surface forces should be taken into account. Then the shape of the equilibrium surface Γ is defined not by (2.3) and (2.4), but by the Laplace condition for the pressure jump:

$$(P_0)_\Gamma - p_a = -\sigma(k_1 + k_2) \quad (\text{on } \Gamma), \tag{2.5}$$

where σ is the positive surface tension coefficient at the boundary between fluid and gas, while k_1 and k_2 are the principal curvatures of the surface Γ.

From (2.1) and (2.5), we see that

$$-\sigma(k_1 + k_2) = -\rho g x_3 + \frac{1}{2}\,\rho\omega_0^2(x_1^2 + x_2^2) + c - p_a. \tag{2.6}$$

This is a nonlinear partial differential equation for the function $x_3 = f(x_1, x_2)$ defining the equation of the free surface Γ. The following boundary-value condition (the Dupré–Young condition) should be satisfied at its boundary $\partial\Gamma$:

$$\sigma \cos\delta = \sigma_1 - \sigma_0; \tag{2.7}$$

here δ is the contact angle (wetting angle), $0 \le \delta \le \pi$, σ_1 is the nonnegative surface tension coefficient at the boundary between solid body and gas, and σ_0 is the same coefficient at the boundary between fluid and solid body.

Equation (2.6) and conditions (2.7) and (2.4) allow us to find the shape of the domain Ω occupied by the fluid (in particular, the equation of the equilibrium surface Γ) by the given volume V of the fluid, characteristics of the three media σ, σ_1, and σ_0 along the contact line $\partial\Gamma$, and gravitational field g. In general, it is a complicated nonlinear problem. If the vessel containing the fluid has an axial symmetry with respect to the rotation axis, then the problem to find an equilibrium state of the system is much simpler. It is explained in detail in the initial sections of [19–21].

Assuming that the static problem is resolved, consider motions of a fluid in a vessel, close to the solid-state rotation. Represent the pressure $P(t, x)$ of the fluid as

$$P(t, x) = P_0(x) + p(t, x), \quad x = (x_1, x_2, x_3) \in \Omega, \tag{2.8}$$

where $p(t, x)$ is the dynamic pressure. Then we have the following system of linearized Navier–Stokes equations (see, e.g., [16, pp. 313, 356–357]) for the sought field $\vec{u}(t, x)$ describing the motion of the fluid in the coordinate system $Ox_1x_2x_3$ rigidly connected with the uniformly rotating vessel and for the dynamic pressure $p(t, x)$:

$$\frac{\partial \vec{u}}{\partial t} - 2\omega_0 \vec{u} \times \vec{e}_3 + \frac{1}{\rho}\nabla p = \nu\Delta\vec{u} + \vec{f}, \quad \operatorname{div}\vec{u} = 0 \quad (\text{in } \Omega); \tag{2.9}$$

here ν is the positive coefficient of the kinematic viscosity of the fluid, while $\vec{f} = \vec{f}(t, x)$ is the small field of external forces superposed with the gravitational field $-\vec{g}\vec{e}_3$. Let S denote the solid wall of the vessel Ω, contacting the fluid in the relative equilibrium state. Then the boundary $\partial\Omega$ consists of S and Γ (more exactly, of their closures), while the following adhesion condition should be satisfied at S:

$$\vec{u} = \vec{0} \quad (\text{on } S). \tag{2.10}$$

It is convenient to use the curvilinear coordinate system $\widetilde{O}\xi^1\xi^2\xi^3$ to write the kinematic and dynamic conditions on Γ. Then the equation of Γ has the form $\xi^3 = 0$, the Lamé coefficient $h_3\big|_\Gamma$ is identically equal to 1, and the normal vector $\vec{n}$ is directed outside Ω. Then, assuming that

$$\xi^3 = \zeta(t, \widehat{\xi}), \quad \widehat{\xi} := (\xi^1, \xi^2) \in \Gamma, \tag{2.11}$$

is the equation of the free moving surface $\widetilde{\Gamma}(t)$, we obtain the kinematic condition

$$\frac{\partial \zeta}{\partial t} = u_n := \vec{u} \cdot \vec{n} \quad (\text{on } \Gamma). \tag{2.12}$$

The dynamic conditions on Γ mean that the tangential tensions are equal to zero and the normal tension equals the pressure jump arising due to gravitational, capillary, and centrifugal forces. If $u_{i,k}$ denotes the covariant derivative of the covariant vector u_i with respect to the variable ξ^k, then the specified conditions on Γ read as

$$\rho\nu(u_{i,3} + u_{3,i}) = 0, \quad i = 1, 2, \tag{2.13}$$

$$-p + 2\rho\nu u_{3,3} = -\mathcal{L}_\sigma\zeta := \sigma\Delta_\Gamma\zeta - a_\Gamma\zeta, \tag{2.14}$$

$$a_\Gamma := -\sigma(k_1^2 + k_2^2) + \rho g \cos(\widehat{\vec{n}, \vec{e}_3}) - \rho\omega_0^2 r \cos(\widehat{\vec{n}, \vec{e}_r}), \tag{2.15}$$

where Δ_Γ is the Laplace–Beltrami operator acting on functions defined on Γ, $r = \sqrt{x_1^2 + x_2^2}$, and $\vec{e}_r$ is the unit vector of the axis Or of the cylindrical coordinate system $Or\theta x_3$ introduced according to the Cartesian coordinate system $Ox_1x_2x_3$.

Two more relations and initial-value conditions are needed to complete the posing of the initial-boundary value problem corresponding to small motions of the rotating capillary fluid. Since not only velocities, but displacements of the fluid particles are equal to zero at the solid wall S, we assume by continuity that the following condition is satisfied at the contour $\partial\Gamma$, i.e., at the intersection of the surfaces Γ and S:

$$\zeta(t, \widehat{\xi}) = 0 \quad (\text{on } \partial\Gamma). \tag{2.16}$$

Also, the function $\zeta(t, x)$ should satisfy the condition of the volume preservation under oscillations. This condition is obtained by variation of relation (2.4) and takes the form

$$\int_\Gamma \zeta \, d\Gamma = 0. \tag{2.17}$$

Finally, the following initial-value conditions should be posed at $t = 0$:

$$\vec{u}(0, x) = \vec{u}^0(x), \quad x \in \Omega; \quad \zeta(0, \widehat{\xi}) = \zeta^0(\widehat{\xi}), \quad \widehat{\xi} \in \Gamma. \tag{2.18}$$

Thus, the sought functions $\vec{u}(t, x)$, $p(t, x)$, and $\zeta(t, \widehat{\xi})$ should satisfy Eqs. (2.9), conditions (2.10), (2.12)–(2.16) (boundary-value conditions), integral link (2.17), and conditions (2.18) (initial-value conditions). It is assumed that the physical parameters of the system, the function $a_\Gamma(\widehat{\xi})$ from (2.15) and the functions $\vec{f}(t, x)$, $\vec{u}^0(x)$, and $\zeta^0(\widehat{\xi})$ are given. The problem is considered in a domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ consisting of two parts: an infinitely differentiable equilibrium surface Γ and a Lipschitz solid wall S. Also, we assume that the dihedral angle δ between Γ and S (this angle is constant by virtue of (2.7)) satisfies the condition

$$0 < \delta < \pi. \tag{2.19}$$

Hence, the whole boundary $\partial\Omega$ of the domain $\Omega \subset \mathbb{R}^3$ is Lipschitz. Note that the function $a_\Gamma(\widehat{\xi})$ from (2.15) is continuous in the closed domain $\overline{\Gamma}$ because Γ is sufficiently smooth.

2.2. Basic function spaces. We will use methods of functional analysis and theory of partial differential equations. Introduce necessary function spaces (see [16, Secs. 2.1 and 2.2]).

1. The space $\vec{L}_2(\Omega)$ with the norm (and the corresponding scalar product) introduced as follows:

$$\|\vec{u}\|^2_\Omega := \int_\Omega |\vec{u}(x)|^2 \, d\Omega \tag{2.20}$$

If $\vec{u}(x)$ is the field of velocities of the fluid in the domain Ω, then fields of velocities with finite kinetic energies correspond to elements $\vec{u} \in \vec{L}_2(\Omega)$.

Further, we need the the following orthogonal decomposition of $\vec{L}_2(\Omega)$:

$$\vec{L}_2(\Omega) = \vec{G}_{0,\Gamma}(\Omega) \oplus \vec{J}_{0,S}(\Omega), \tag{2.21}$$

$$\vec{G}_{0,\Gamma}(\Omega) := \{\vec{w} \in \vec{L}_2(\Omega) : \quad \vec{w} = \nabla\varphi, \quad \varphi = 0 \ (\text{on } \Gamma)\}, \tag{2.22}$$

$$\vec{J}_{0,S}(\Omega) := \{\vec{v} \in \vec{L}_2(\Omega) : \quad \operatorname{div} \vec{v} = 0 \ (\text{in } \Omega), \quad v_n := \vec{v} \cdot \vec{n} = 0 \ (\text{on } S)\}. \tag{2.23}$$

Here the operations $\operatorname{div} \vec{v}$ and v_n are understood in the sense of distributions (see [16, pp. 100–102]).

Note that all elements from the subspace $\vec{J}_{0,S}(\Omega)$ describe velocity fields of the ideal fluid in an open vessel.

2. For motions of a homogeneous viscous fluid in a vessel, the stress tensor caused by viscosity forces has the components

$$\mu\tau_{ij} := \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2, 3, \tag{2.24}$$

where $\mu = \rho\nu > 0$ is the dynamic viscosity coefficient. The viscosity forces lead to energy dissipation; its velocity is computed as follows:

$$\mu E(\vec{u}, \vec{u}) := \frac{1}{2}\,\mu \int\limits_{\Omega} \sum_{i,j=1}^{3} |\tau_{ij}(\vec{u})|^2 \, d\Omega. \tag{2.25}$$

The set of solenoidal vector fields satisfying the adhesion condition at the solid wall S and the condition of a finite velocity of the energy dissipation in a viscous fluid, i.e., the set

$$\vec{J}^{\,1}_{0,S}(\Omega) := \{\vec{u} \in \vec{L}_2(\Omega) : \quad E(\vec{u}, \vec{u}) < \infty, \quad \operatorname{div} \vec{u} = 0 \ (\text{in } \Omega), \quad \vec{u} = \vec{0} \ (\text{on } S)\}, \tag{2.26}$$

is a subspace in the space $\vec{H}^1(\Omega)$, where any component $u_i(x)$ belongs to $H^1(\Omega)$. The norm in this subspace is introduced as follows:

$$\|\vec{u}\|^2_{1,\Omega} := E(\vec{u}, \vec{u}). \tag{2.27}$$

This norm is equivalent to the standard norm of the space $\vec{H}^1(\Omega)$.

Note that the subspace $\vec{J}^{\,1}_{0,S}(\Omega)$ is dense in $\vec{J}_{0,S}(\Omega)$ (see [16, c. 112–115]) and $\vec{J}^{\,1}_{0,S}(\Omega)$ is compactly imbedded in $\vec{J}_{0,S}(\Omega)$, i.e., the imbedding operator from $\vec{J}^{\,1}_{0,S}(\Omega)$ to $\vec{J}_{0,S}(\Omega)$ is compact.

3. The space $L_2(\Gamma)$, where the norm is introduced by the relation

$$\|\varphi\|^2_0 := \int\limits_{\Gamma} |\varphi(\widehat{\xi})|^2 \, d\Gamma \tag{2.28}$$

and its subspace $L_{2,\Gamma}$ (its codimension is equal to 1) consisting of the elements of $L_2(\Gamma)$ orthogonal to the unit function 1_Γ:

$$L_{2,\Gamma} := \{\varphi \in L_{2,\Gamma} : \quad (\varphi, 1_\Gamma)_0 = 0\}. \tag{2.29}$$

2.3. Equipped Hilbert spaces. Also, we need equipments of the spaces introduced above (see [2, Sec. 1.1]). Since norm (2.25) is equivalent to the norm of $\vec{H}^1(\Omega)$ for elements from $\vec{J}^{\,1}_{0,S}(\Omega)$, we have the following inequalities:

$$0 < c_1 \le \frac{\|\vec{u}\|^2_{1,\Omega}}{\|\vec{u}\|^2_{\vec{H}^1(\Omega)}} \le c_2 < \infty, \tag{2.30}$$

and, therefore,

$$\|\vec{u}\|^2_{1,\Omega} \ge c_1 \|\vec{u}\|^2_{\vec{H}^1(\Omega)} \ge c_1 \|\vec{u}\|^2_{\Omega} \quad \forall \vec{u} \in \vec{J}^{\,1}_{0,S}(\Omega). \tag{2.31}$$

This implies that $\vec{J}^{\,1}_{0,S}(\Omega)$ and $\vec{J}_{0,S}(\Omega)$ form a Hilbert pair (see [16, pp. 32–34]) and the generator A of that pair form a scale of function spaces E^α, $\alpha \in \mathbb{R}$, such that

$$E^0 = \vec{J}_{0,S}(\Omega), \quad E^{1/2} = \vec{J}^{\,1}_{0,S}(\Omega) = \mathcal{D}(A^{1/2}), \tag{2.32}$$

and

$$\|\vec{u}\|^2_{1,\Omega} = \|A^{1/2}\vec{u}\|^2_{\Omega} \quad \forall \vec{u} \in \vec{J}^{\,1}_{0,S}(\Omega). \tag{2.33}$$

Finally, using the spaces $\vec{J}_{0,S}(\Omega)$ and $\vec{J}^{\,1}_{0,S}(\Omega)$, we create the equipment

$$\vec{J}^{\,1}_{0,S}(\Omega) \hookrightarrow \vec{J}_{0,S}(\Omega) \hookrightarrow (\vec{J}^{\,1}_{0,S}(\Omega))^* \tag{2.34}$$

of the space $\vec{J}_{0,S}(\Omega)$. Then any linear bounded functional with respect to $\vec{u} \in \vec{J}^{\,1}_{0,S}(\Omega)$ is represented as a "scalar product" in $\vec{J}_{0,S}(\Omega)$, i.e.,

$$l_{\vec{\eta}}(\vec{u}) = \langle \vec{u}, \vec{\eta} \rangle_\Omega, \quad \vec{u} \in \vec{J}^{\,1}_{0,S}(\Omega), \quad \vec{\eta} \in (\vec{J}^{\,1}_{0,S}(\Omega))^*, \tag{2.35}$$

$$|l_{\vec{\eta}}(\vec{u})| \le \|\vec{u}\|_{1,\Omega} \|\vec{\eta}\|_{(\vec{J}^{\,1}_{0,S}(\Omega))^*}. \tag{2.36}$$

Here $\langle \vec{u}, \vec{\eta} \rangle_\Omega$ is the extension (a closure by continuity) of the scalar product $(\vec{u}, \vec{\eta})_\Omega$ for $\vec{u} \in \vec{J}^{\,1}_{0,S}(\Omega)$ corresponding to passing from elements $\vec{\eta} \in \vec{J}_{0,S}(\Omega)$ to elements $\vec{\eta} \in (\vec{J}^{\,1}_{0,S}(\Omega))^*$.

Introduce the normal trace operator γ_n:

$$\gamma_n \vec{u} := (\vec{u} \cdot \vec{n})_\Gamma = (u_n)_\Gamma \quad \forall \vec{u} \in \vec{J}^1_{0,S}(\Omega). \tag{2.37}$$

Since $\vec{J}^1_{0,S}(\Omega)$ is a subspace of $\vec{H}^1(\Omega)$, Ω is a Lipschitz domain, and Γ is a sufficiently smooth part of the boundary $\partial\Omega$, it follows from the Gagliargo theorem (see [4]) that γ_n is a bounded operator from $\vec{J}^1_{0,S}(\Omega)$ to the space $H^{1/2}(\Gamma)$, where the norm is defined as follows:

$$\|\varphi\|^2_{H^{1/2}(\Gamma)} := \int_\Gamma |\varphi|^2 \, d\Gamma + \int_\Gamma \int_\Gamma \frac{|\varphi(\widehat{\xi}) - \varphi(\widehat{\eta})|^2}{|\widehat{\xi} - \widehat{\eta}|^3} \, d\Gamma_{\widehat{\xi}} \, d\Gamma_{\widehat{\eta}}. \tag{2.38}$$

Also, note that

$$\int_\Gamma (\gamma_n \vec{u}) \, d\Gamma = 0 \tag{2.39}$$

for $\vec{u} \in \vec{J}^1_{0,S}(\Omega)$, i.e., $\gamma_n \vec{u} \in L_{2,\Gamma}$. Therefore, γ_n is a bounded operator from $\vec{J}^1_{0,S}(\Omega)$ to the space

$$H^{1/2}_\Gamma := H^{1/2}(\Gamma) \cap L_{2,\Gamma}, \tag{2.40}$$

which is densely imbedded in $L_{2,\Gamma}$. This implies that $H^{1/2}_\Gamma$ and $L_{2,\Gamma}$ form a Hilbert pair of spaces; using it, we create the equipment

$$H^{1/2}_\Gamma \hookrightarrow L_{2,\Gamma} \hookrightarrow (H^{1/2}_\Gamma)^*. \tag{2.41}$$

Correspondingly, any functional bounded over $H^{1/2}_\Gamma$ is representable as

$$l_\psi(\varphi) := \langle \varphi, \psi \rangle_0, \quad \forall \varphi \in H^{1/2}_\Gamma, \quad \psi \in (H^{1/2}_\Gamma)^*, \tag{2.42}$$

$$|\langle \varphi, \psi \rangle_0| \le \|\varphi\|_{H^{1/2}_\Gamma} \|\psi\|_{(H^{1/2}_\Gamma)^*}, \tag{2.43}$$

where $\langle \varphi, \psi \rangle_0$ is the extension of the functional $(\varphi, \psi)_0$ by continuity for $\varphi \in H^{1/2}_\Gamma$, $\psi \in (H^{1/2}_\Gamma)^*$.

2.4. Abstract Green formula for a triple of Hilbert spaces. Auxiliary Stokes problems. We will use an abstract Green formula for a triple of Hilbert spaces and its particular case corresponding to spaces and equipments introduced in Secs. 2.2 and 2.3. The corresponding general result from [12] (see also [11, 15] and [22, pp. 187–189]) is given below.

Theorem 2.1. *Let E, F, and G be Hilbert spaces such that the following conditions are satisfied:*

(1) *The space F is densely imbedded in E, i.e., F is dense in E and*

$$\|u\|_E \le a\|u\|_F, \quad \forall u \in F. \tag{2.44}$$

(2) *A trace operator γ is defined in the space F such that it is bounded as an operator from F to G, γ maps F to a dense set $\mathcal{R}(\gamma) =: G_+ \subset G$, and*

$$\|\varphi\|_G \le b\|\varphi\|_{G_+} \quad \forall \varphi \in G_+. \tag{2.45}$$

Then there exist operators L and ∂ defined on F and bounded as operators from $\mathcal{D}(L) = \mathcal{D}(\partial) = F$ to F^* and $G_- = (G_+)^*$, respectively, such that the abstract Green formula is valid:

$$\langle \eta, Lu \rangle_E = (\eta, u)_F - \langle \gamma\eta, \partial u \rangle_G \quad \forall \eta, u \in F. \tag{2.46}$$

The angular brackets denote the corresponding functionals, $Lu \in F^*$, and $\partial u \in (G_+)^*$. The operators L and ∂ are uniquely defined by the spaces E, F, and G, their scalar products, and the operator γ.

Consider the kernel of the operator γ, i.e., the set $N := \operatorname{Ker}\gamma \subset F$, and its orthogonal complement

$$M = F \ominus N. \tag{2.47}$$

Then $\gamma_M = \gamma\big|_M$ is a one-to-one map of M to G_+. Introduce the operator T_M adjoint to γ_M in the sense of the scalar product in G:

$$(\eta, T_M\psi)_F = \langle \gamma_M \eta, \psi \rangle_G \quad \forall \eta \in M, \quad \forall \psi \in (G_+)^*. \tag{2.48}$$

Assume that the operator A of the Hilbert pair $(F; E)$ is defined on F. Then $\mathcal{D}(A) = F$, $\mathcal{R}(A) = F^*$, and

$$(\eta, u)_F = \langle \eta, Au \rangle_E \quad \forall \eta, u \in F. \tag{2.49}$$

Consider the abstract Neumann boundary-value problem for the Poisson equation:

$$Lu = f, \quad \partial u = \psi. \tag{2.50}$$

Theorem 2.2 (see [12, Theorem 2]). *Problem* (2.50) *has a unique weak solution* $u \in F$ *if and only if the following conditions are satisfied*:

$$f \in F^*, \quad \psi \in (G_+)^*. \tag{2.51}$$

That solution has the form

$$u = A^{-1}f + T_M\psi, \tag{2.52}$$

where $A\colon F \to F^*$ *is the generator of the Hilbert pair* $(F; E)$ *and* $T_M = (\gamma_M)^*$. □

Remark 2.1. It follows from (2.52) that the solution of problem (2.50) is representable as

$$u = v + w, \tag{2.53}$$

where $v = A^{-1}f$ is the weak solution of the abstract Neumann problem for the homogeneous "boundary-value" condition, i.e.,

$$Lv = f, \quad \partial v = 0, \tag{2.54}$$

while $w = T_M\psi$ is the weak solution of the abstract Neumann problem for the homogeneous equation:

$$Lw = 0, \quad \partial w = \psi. \tag{2.55}$$

The specified weak solutions are defined by the following identities respectively:

$$(\eta, v)_F = \langle \eta, f \rangle_E \quad \forall \eta \in F, \tag{2.56}$$

$$(\eta, w)_F = \langle \gamma\eta, \psi \rangle_G \quad \forall \eta \in F. \tag{2.57}$$

Using the facts provided above, consider the triple of spaces $E = \vec{J}_{0,S}(\Omega)$, $F = \vec{J}^1_{0,S}(\Omega)$, and $G = L_2(\Gamma) \oplus L_2(\Gamma) \oplus L_{2,\Gamma}$ as well as the trace operator γ:

$$\gamma\vec{u} := \vec{u}|_\Gamma \quad \forall \vec{u} \in \vec{J}^1_{0,S}(\Omega). \tag{2.58}$$

Let us check whether conditions (1) and (2) of Theorem 2.1 are fulfilled for the above objects. Note that

$$\gamma_n\vec{u} = (\vec{u} \cdot \vec{n})\,|_\Gamma = u_3|_\Gamma, \tag{2.59}$$

where $\widetilde{O}\xi^1\xi^2\xi^3$ is the local coordinate system introduced in a neighborhood of Γ (see Sec. 2.1).

Since the space $\vec{J}^1_{0,S}(\Omega)$ is dense in $\vec{J}_{0,S}(\Omega)$ and inequality (2.31) holds, it follows that $\vec{J}^1_{0,S}(\Omega)$ and $\vec{J}_{0,S}(\Omega)$ form a Hilbert pair of spaces, i.e., condition (1) is fulfilled.

Theorem 2.3 (see [12, Theorem 5]). *The operator A of the Hilbert pair $(\vec{J}^1_{0,S}(\Omega); \vec{J}_{0,S}(\Omega))$ is the operator of the boundary-value problem*

$$A\vec{v} := -P_{0,S}\Delta\vec{v} + \nabla p_v = \vec{f}, \quad \operatorname{div} \vec{v} = 0 \ (in\, \Omega), \tag{2.60}$$

$$\vec{v} = \vec{0} \ (on\ S), \quad \tau_{i3}(\vec{v}) - p_v\delta_{i3} = 0 \ (on\, \Gamma), \quad i = 1, 2, 3, \tag{2.61}$$

$$\Delta p_v = 0 \ (in\, \Omega), \quad \frac{\partial p_v}{\partial n} = 0 \ (on\ S). \tag{2.62}$$

Here $P_{0,S} : \vec{L}_2(\Omega) \to \vec{J}_{0,S}(\Omega)$ is an orthogonal projector on the subspace $\vec{J}_{0,S}(\Omega)$, while $\tau_{ij}(\vec{v})$, unlike denotation (2.24)*, is expressed in the curvilinear coordinate system $\widetilde{O}\xi^1\xi^2\xi^3$ via covariant derivatives ξ^j:*

$$\tau_{ij}(\vec{v}) := v_{i,j} + v_{j,i}, \quad i, j = 1, 2, 3. \tag{2.63}$$

If problem (2.60)–(2.62) *has a solution $\vec{v} \in \vec{H}^2(\Omega) \cap \vec{J}^1_{0,S}(\Omega)$, then*

$$\nabla p_v \in \vec{G}_{h,S}(\Omega) := \left\{ \vec{u} \in \vec{J}_{0,S}(\Omega) : \vec{u} = \nabla p,\ \Delta p = 0 \ (in\, \Omega),\ \frac{\partial p}{\partial n} = 0 \ (on\ S),\ \int_\Gamma p\, d\Gamma = 0 \right\}. \tag{2.64}$$

For any $\vec{f} \in \vec{J}_{0,S}(\Omega)$, problem (2.60)–(2.62) *has a unique generalized solution $\vec{v} = A^{-1}\vec{f} \in \mathcal{D}(A) \subset \vec{J}^1_{0,S}(\Omega) = \mathcal{D}(A^{1/2})$. For any $\vec{f} \in (\vec{J}^1_{0,S}(\Omega))^*$, the specified problem has a unique weak solution $\vec{v} \in \vec{J}^1_{0,S}(\Omega)$. Conversely, for any element $\vec{v} \in \vec{J}^1_{0,S}(\Omega)$, there exists $\vec{f} \in (\vec{J}^1_{0,S}(\Omega))^*$ such that the specified element is a weak solution of problem* (2.60)–(2.62).

Remark 2.2. Generalized solutions of problem (2.60)–(2.62) are defined by the identity

$$E(\vec{\eta}, \vec{v}) = (\vec{\eta}, \vec{f})_\Omega \quad \forall \vec{\eta} \in \vec{J}^1_{0,S}(\Omega), \tag{2.65}$$

while its weak solutions are defined by the identity

$$E(\vec{\eta}, \vec{v}) = \langle \vec{\eta}, \vec{f} \rangle_\Omega \quad \forall \vec{\eta} \in \vec{J}^1_{0,S}(\Omega). \tag{2.66}$$

To check condition (2) of Theorem 2.1, we note that, by virtue of the trace theorem for domains with Lipschitz boundaries (see [4]), the trace operator (2.58) is bounded as an operator from $\vec{J}^1_{0,S}(\Omega) \subset \vec{H}^1(\Omega)$ to the space

$$G_+ := H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \times H^{1/2}_\Gamma, \quad H^{1/2}_\Gamma := H^{1/2}(\Gamma) \cap L_{2,\Gamma}, \tag{2.67}$$

which is dense in the space

$$G := L_2(\Gamma) \oplus L_2(\Gamma) \oplus L_{2,\Gamma}; \tag{2.68}$$

moreover, G_+ is compactly imbedded in G and

$$\|\vec{\varphi}\|_G \le b\|\vec{\varphi}\|_{G_+} \quad \forall \vec{\varphi} \in G_+. \tag{2.69}$$

This implies that the spaces G_+ and G form a Hilbert pair. Therefore, condition (2) of Theorem 2.1 is fulfilled.

Thus, Theorem 2.1 holds for the triple of spaces $E = \vec{J}_{0,S}(\Omega)$, $F = \vec{J}^1_{0,S}(\Omega)$, and $G = L_2(\Gamma) \oplus L_2(\Gamma) \oplus L_{2,\Gamma}$ and the trace operator γ from (2.58). This means that the following assertion is valid.

Theorem 2.4 (see [12, Theorem 9]). *Let a domain Ω satisfy the above assumptions. Then the abstract Green formula*

$$\langle \vec{\eta}, -P_{0,S}\Delta\vec{u} + \nabla p_u \rangle_\Omega = E(\vec{\eta}, \vec{u}) - \sum_{i=1}^{3} \langle (\gamma\vec{\eta})^i, (\tau_{i3}(\vec{u}) - p_u\delta_{i3}) \rangle_0 \tag{2.70}$$

holds for any functions $\vec{\eta}$ and $\vec{u}$ from $\vec{J}^1_{0,S}(\Omega)$, where $\tau_{i3}(\vec{u})$ are defined by relations (2.63) *and*

$$-P_{0,S}\Delta\vec{u}+\nabla p_u = L\vec{u}\in\left(\vec{J}^1_{0,S}(\Omega)\right)^*, \quad \gamma\vec{\eta}=\vec{\eta}\big|_\Gamma\in G_+, \tag{2.71}$$

$$\partial\vec{u}=\sum_{i=1}^{3}\left(\tau_{i3}(\vec{u})-p_u\delta_{i3}\right)\vec{e}_{\xi_i}\in(G_+)^*.$$

Green formula (2.70) is directly related to the Stokes boundary-value problem:

$$-P_{0,S}\Delta\vec{u}+\nabla p_u=\vec{f}, \quad \operatorname{div}\vec{u}=0 \text{ (in } \Omega), \quad \vec{u}=\vec{0} \text{ (on } S), \tag{2.72}$$

$$\partial\vec{u}=\sum_{i=1}^{3}\left(\tau_{i3}(\vec{u})-p_u\delta_{i3}\right)\vec{e}_{\xi_i}=\vec{\psi} \text{ (on } \Gamma), \tag{2.73}$$

$$\Delta p_u=0 \text{ (in } \Omega), \quad \frac{\partial p_u}{\partial n}=0 \text{ (on } S), \quad \int_\Gamma p_u\,d\Gamma=0. \tag{2.74}$$

This problem arises in the investigation of small motions of viscous fluid in an open vessel (see (2.9)–(2.18)).

Theorem 2.5 (see [12, Theorem 8])**.** *If the conditions*

$$\vec{f}\in(\vec{J}^1_{0,S}(\Omega))^*, \quad \vec{\psi}\in(G_+)^* \tag{2.75}$$

are satisfied, then problem (2.72)*–*(2.74) *has a unique solution $\vec{u}\in\vec{J}^1_{0,S}(\Omega)$ representable as follows*:

$$\vec{u}=\vec{v}+\vec{w}, \quad \nabla p_u=\nabla p_v+\nabla p_w, \tag{2.76}$$

where $\vec{v}$ is the weak solution of problem (2.60)*–*(2.62) (*the first Kreĭn auxiliary problem*)*, while $\vec{w}$ is the weak solution of the following problem* (*the second Kreĭn auxiliary problem*):

$$-P_{0,S}\Delta\vec{w}+\nabla p_w=\vec{0}, \quad \operatorname{div}\vec{w}=0 \ (in\ \Omega), \tag{2.77}$$

$$\vec{w}=\vec{0}\ (on\ S), \quad \partial\vec{w}=\vec{\psi}\ (on\ \Gamma), \tag{2.78}$$

$$\Delta p_w=0\ (in\ \Omega), \quad \frac{\partial p_w}{\partial n}=0\ (on\ S), \quad \int_\Gamma p_w\,d\Gamma=0. \tag{2.79}$$

Conversely, for any field $\vec{u}$ from $\vec{J}^1_{0,S}(\Omega)$, there exist $\vec{f}$ and $\vec{\psi}$ satisfying conditions (2.75) *such that the specified field is representable by* (2.76)*, where $\vec{v}$ and $\vec{w}$ are the weak solutions of problems* (2.60)*–*(2.62) *and* (2.77)*–*(2.79)*, respectively.*

Remark 2.3. In Green formula (2.70) for Stokes problem (2.72)–(2.74), the field p_u is linked with $\vec{u}$ (see (2.76)). Sometimes, it is written in form where the field p does not depend on $\vec{u}$. This means that p_u+p is substituted for p_u in (2.70) and (2.71).

Indeed, let $\nabla p\in\vec{G}_{h,S}(\Omega)\subset\vec{J}_{0,S}(\Omega)$ (see (2.64)). Then

$$\langle\vec{\eta},\nabla p\rangle_\Omega=(\vec{\eta},\nabla p)_\Omega=\int_\Gamma \eta_n\,(\gamma p)\,d\Gamma=-\langle\gamma\eta_3,-p\rangle_0. \tag{2.80}$$

The left-hand side is defined not only for $\nabla p\in\vec{J}_{0,S}(\Omega)$, but for $\nabla p\in(\vec{J}^1_{0,S}(\Omega))^*$. Therefore, the right-hand side of (2.80) and relation (2.70) (with the corresponding changes) have a sense for $\nabla p\in(\vec{J}^1_{0,S}(\Omega))^*$ as well.

2.5. Systems of differential-operator equations. Consider initial-boundary value problem (2.9)–(2.18) again and find a system of differential-operator equations in a Hilbert space, associated with that problem.

First of all, it follows from (2.17) that $\zeta \in L_{2,\Gamma}$ for any $t \geq 0$. Hence, $P_\Gamma \zeta = \zeta$, where $P_\Gamma : L_2(\Gamma) \to L_{2,\Gamma}$ is an orthogonal projector on $L_{2,\Gamma}$:

$$P_\Gamma \varphi := \varphi - |\Gamma|^{-1} \int_\Gamma \varphi \, d\Gamma \quad \forall \varphi \in L_2(\Gamma). \tag{2.81}$$

Introduce the operator B_σ:

$$B_\sigma \zeta := P_\Gamma \mathcal{L}_\sigma P_\Gamma \zeta, \quad \zeta \in \mathcal{D}(B_\sigma) = H_0^2(\Gamma) \cap L_{2,\Gamma}, \tag{2.82}$$

where the expression $\mathcal{L}_\sigma$ is defined in (2.14) and (2.15), while condition (2.16) is taken into account by means of the choice of $\mathcal{D}(B_\sigma)$.

Lemma 2.1 (see [16, pp. 163–164]). *The operator* $B_\sigma : \mathcal{D}(B_\sigma) \subset L_{2,\Gamma} \to L_{2,\Gamma}$ *is unbounded, self-adjoint, and bounded from below. Its quadratic form is representable as follows*:

$$(B_\sigma \zeta, \zeta)_0 = \int_\Gamma \left[|\nabla_\Gamma \zeta|^2 + a_\Gamma |\zeta|^2 \right] d\Gamma, \quad \zeta \in \mathcal{D}(B_\sigma). \tag{2.83}$$

If B_σ *is positive definite in* $L_{2,\Gamma}$, *then quadratic form* (2.83) *is equivalent to the standard norm of the space* $H^1(\Gamma)$ *and to the norm*

$$\|\zeta\|_\nabla^2 := \int_\Gamma |\nabla_\Gamma \zeta|^2 \, d\Gamma, \quad \zeta \in H_0^1(\Gamma) \cap L_{2,\Gamma}. \tag{2.84}$$

The operator B_σ *has a discrete real spectrum* $\{\lambda_k(B_\sigma)\}_{k=1}^\infty$, *the multiplicity of any its eigenvalue is finite,* $+\infty$ *is the limit point of its eigenvalues, and the system of its eigenelements* $\{\zeta_k(B_\sigma)\}_{k=1}^\infty$ *forms an orthogonal basis in* $L_{2,\Gamma}$.

Definition 2.1. We say that a relative equilibrium state of a rotating capillary viscous fluid is statically stable with respect to the linearizarion if the operator B_σ is positive definite in $L_{2,\Gamma}$, i.e.,

$$(B_\sigma \zeta, \zeta)_0 \geq c\|\zeta\|_0^2, \quad c > 0, \quad \zeta \in \mathcal{D}(B_\sigma). \tag{2.85}$$

Since quadratic form $(B_\sigma \zeta, \zeta)_0$ (see (2.83)) is equal to the doubled potential energy of the given hydraulic system, it follows that condition (2.85) is satisfied if and only if the potential energy of the system has a rough minimum at a relative equilibrium state. Also, note that the operator B_σ is invertible under assumption (2.85), the inverse operator B_σ^{-1} is positive and compact, and all eigenvalues $\lambda_k(B_\sigma)$ of the operator B_σ are positive.

Using the operator B_σ introduced in (2.82), we find another form of condition (2.14). Note that the pressure $p(t,x)$ can be defined up to an arbitrary function of t because internal forces in the fluid are determined by the pressure gradients ∇p. Therefore, we may assume that an additional norming condition is satisfied:

$$\int_\Gamma p \, d\Gamma = 0. \tag{2.86}$$

Apply the orthogonal projector P_Γ to both sides of (2.14) and use the relation

$$\int_\Gamma u_{3,3} \, d\Gamma = 0 \tag{2.87}$$

(we prove it in the same way as in [16, p. 115], where it is proved for a horizontal Γ). We obtain the boundary-value condition

$$-p + 2\rho\nu u_{3,3} = -B_\sigma \zeta \quad (\text{on } \Gamma) \tag{2.88}$$

instead of (2.14).

Our aim is to obtain a system of differential relations from (2.9)–(2.18). Then we obtain the Cauchy problem for a differential-operator equation in a Hilbert space from those differential relations.

Assume that all terms of Eq. (2.9) are functions of the variable t valued in $\vec{J}_{0,S}(\Omega)$. More exactly, $\vec{u}(t,x)$ is a function of the variable t valued in $\vec{J}_{0,S}(\Omega)$ and $\nabla p(t,x)$ is a function of the variable t valued in $\vec{G}(\Omega) := \vec{G}_{h,S}(\Omega) \oplus \vec{G}_{0,\Gamma}(\Omega)$ (see (2.21)–(2.23) and (2.64)). Then, applying the orthogonal projectors $P_{0,\Gamma}$ and $P_{0,S}$ on the subspaces $\vec{G}_{0,\Gamma}(\Omega)$ and $\vec{J}_{0,S}(\Omega)$ to both sides of (2.9), we obtain the relations

$$\frac{1}{\rho}\nabla\varphi = 2\omega_0 P_{0,\Gamma}\left(\vec{u}\times\vec{e}_3\right) + \nu P_{0,\Gamma}\Delta\vec{u} + P_{0,\Gamma}\vec{f}, \quad \nabla\varphi := P_{0,\Gamma}\nabla p, \tag{2.89}$$

$$-\nu P_{0,S}\Delta\vec{u} + \frac{1}{\rho}\nabla\widetilde{p} = -\frac{d\vec{u}}{dt} + 2\omega_0 P_{0,S}\left(\vec{u}\times\vec{e}_3\right) + P_{0,S}\vec{f}, \quad \nabla\widetilde{p} := P_{0,S}\nabla p. \tag{2.90}$$

It follows from (2.89) that $\nabla\varphi$ is directly computed via the known field $\vec{f}$ and the field $\vec{u}$ found from (2.90). On the other hand, φ is not included in (2.90) and (2.10)–(2.18) (the initial-value and boundary-value conditions). Indeed, since

$$p = \widetilde{p} + \varphi, \quad \varphi\big|_\Gamma = 0 \tag{2.91}$$

by virtue of (2.86), (2.89), (2.90), and (2.22), it follows that $p|_\Gamma = \widetilde{p}|_\Gamma$. Hence, condition (2.88) takes the following form:

$$-\widetilde{p} + 2\rho\nu u_{3,3} = -B_\sigma\zeta \text{ (on } \Gamma). \tag{2.92}$$

This reduces problem (2.9)–(2.18) to relation (2.89) and to Stokes problem (2.72)–(2.74), where $\vec{f}$ is replaced by the right-hand side of (2.90) and ψ is replaced by the vector such that its first and second components are equal to zero in the curvilinear coordinate system $\widetilde{O}\xi^1\xi^2\xi^3$ introduced in the neighborhood of Γ (see above), while the third component, i.e., the normal component $\psi_n = \psi_3$, is equal to $-B_\sigma\zeta$ by virtue of (2.92). condition (2.12) (the kinematic condition) should be satisfied as well.

This, Theorem 2.5, and general relation (2.52) (see Theorem 2.2) imply that problem (2.9)–(2.18) (the original problem) is equivalent to relation (2.89) and the following system of equations and initial-value conditions:

$$\vec{u} = \vec{v} + \vec{w}, \quad \nu A\vec{v} = -\frac{d\vec{u}}{dt} + 2\omega_0 P_{0,S}\left(\vec{u}\times\vec{e}_3\right) + P_{0,S}\vec{f}, \tag{2.93}$$

$$\nu\vec{w} = T\left(-B_\sigma\zeta\right), \quad \frac{d\zeta}{dt} = \gamma_n\vec{u}, \quad \vec{u}(0) = \vec{u}^0, \quad \zeta(0) = \zeta^0. \tag{2.94}$$

Here d/dt denotes the derivative with respect to t of a function of the variable t valued in a Hilbert space, while

$$\gamma_n\vec{u} = \left(\vec{u}\cdot\vec{n}\right)_\Gamma, \quad \vec{u} \in \vec{J}^1_{0,S}(\Omega). \tag{2.95}$$

Further, T in (2.94) denotes the restriction of the operator of boundary-value problem (2.77)–(2.79) to the set of elements $\psi \in (G_+)^*$ such that only the third (normal) component of any specified element is different from zero and belongs to the space $(H_\Gamma^{1/2})^*$ (see above). This follows from the fact that any solution $\vec{u}$ of problem (2.9)–(2.18) satisfies conditions (2.13), i.e., by virtue of definition of $\partial\vec{u}$ (see (2.71) and (2.73)), the condition

$$\left(\partial\vec{u}\right)_1 = \left(\partial\vec{u}\right)_2 = 0 \text{ (on } \Gamma) \tag{2.96}$$

holds. Since ∂ is a bounded operator from $F = \vec{J}^1_{0,S}(\Omega)$ to $(G_+)^*$ (see Theorem 2.4), it follows that the set of the specified elements $\vec{u}$ from $\vec{J}^1_{0,S}(\Omega)$ is a subspace of $\vec{J}^1_{0,S}(\Omega)$ containing all solutions of problem (2.9)–(2.18) belonging to $\vec{J}^1_{0,S}(\Omega)$.

Using the first relation of (2.93) and the second relation of (2.94), we can exclude the variables $\vec{u}$ and ζ from (2.93)–(2.94). Differentiate the first relation of (2.94) with respect to t and introduce the operator

$$S_0\vec{u} := -iP_{0,S}\left(\vec{u}\times\vec{e}_3\right). \tag{2.97}$$

This yields the following Cauchy problem for a system of operator-differential equations:

$$\frac{d\vec{v}}{dt}+\frac{d\vec{w}}{dt}+\nu A\vec{v}-2i\omega_0 S_0(\vec{v}+\vec{w})=P_{0,S}\vec{f},\quad \vec{v}(0)=\vec{v}^0, \tag{2.98}$$

$$\frac{d\vec{w}}{dt}+\nu^{-1}TB_\sigma\gamma_n(\vec{v}+\vec{w})=\vec{0},\quad \vec{w}(0)=\vec{w}^0, \tag{2.99}$$

$$\vec{v}^0=\vec{u}^0-\vec{w}^0,\quad \vec{w}^0=-\nu^{-1}TB_\sigma\zeta^0. \tag{2.100}$$

Change the variables:

$$\vec{v}=A^{-1/2}\vec{\xi},\quad \vec{w}=A^{-1/2}\vec{\eta}. \tag{2.101}$$

Formally apply the operator $A^{1/2}$ to both sides of Eqs. (2.98) and (2.99). We obtain the following Cauchy problem from (2.98)–(2.100):

$$\frac{d\vec{\xi}}{dt}+\nu A\vec{\xi}-2i\omega_0 A^{1/2}S_0A^{-1/2}(\vec{\xi}+\vec{\eta})-\nu^{-1}B(\vec{\xi}+\vec{\eta})=A^{1/2}P_{0,S}\vec{f}, \tag{2.102}$$

$$\frac{d\vec{\eta}}{dt}+\nu^{-1}B(\vec{\xi}+\vec{\eta})=\vec{0},\quad \vec{\eta}(0)=-\nu^{-1}Q^*B_\sigma\zeta^0,\quad \vec{\xi}(0)=A^{1/2}\vec{u}^0-\vec{\eta}(0), \tag{2.103}$$

$$B:=Q^*B_\sigma Q,\quad Q:=\gamma_nA^{-1/2},\quad Q^*:=A^{1/2}T. \tag{2.104}$$

We begin to investigate the solvability of the problem on small motions of a capillary viscous fluid partially filling a rotating vessel, i.e., initial-boundary value problem (2.9)–(2.18), starting with problems (2.98)–(2.100) and (2.102)–(2.104).

First we find properties of operator coefficients for those systems of differential equations.

Lemma 2.2. *The operator S_0 defined by* (2.97) *is a self-adjoint bounded operator acting in the space $\vec{J}_{0,S}(\Omega)$ and*

$$\|S_0\|=1,\quad \sigma(S_0)=\sigma_{ess}(S_0)=[-1,1], \tag{2.105}$$

where $\sigma_{ess}(S_0)$ denotes the essential (limit) spectrum of the operator S_0, while $\sigma(S_0)$ denotes its spectrum.

Proof. See [16, Secs. 7.4, 5.1, and 5.3]. □

Lemma 2.3. *The space $\vec{J}^1_{0,S}(\Omega)$ has the orthogonal decomposition*

$$\vec{J}^1_{0,S}(\Omega)=\vec{N}_1(\Omega)\oplus\vec{M}_1(\Omega), \tag{2.106}$$

where

$$\vec{N}_1(\Omega):=\{\vec{v}\in\vec{J}^1_{0,S}(\Omega):\gamma_n\vec{v}=0\ (on\ \Gamma)\}, \tag{2.107}$$

while $\vec{M}_1(\Omega)$ is the subspace of weak solutions of problem (2.77)–(2.79) *for any $\vec{\psi}=(0;0;\psi_3)$, $\psi_3=\psi_n\in(H_\Gamma^{1/2})^*$. The space $\vec{J}_{0,S}(\Omega)=A^{1/2}\vec{J}^1_{0,S}(\Omega)$ has the following orthogonal decomposition*:

$$\vec{J}_{0,S}(\Omega)=\vec{N}_0(\Omega)\oplus\vec{M}_0(\Omega):=A^{1/2}\vec{N}_1(\Omega)\oplus A^{1/2}\vec{M}_1(\Omega). \tag{2.108}$$

Proof. See [16, Secs. 1.3 and 2.2 and p. 310]. □

Lemma 2.4. *The operators*

$$Q=\gamma_nA^{-1/2}:\vec{J}_{0,S}(\Omega)\to L_{2,\Gamma},\quad Q^*=A^{1/2}T:L_{2,\Gamma}\to\vec{J}_{0,S}(\Omega) \tag{2.109}$$

are adjoint to each other and compact, Q is a bounded operator from $\vec{J}_{0,S}(\Omega)$ to $H_\Gamma^{1/2}$, and the extension of Q^* is a bounded operator from $(H_\Gamma^{1/2})^*$ to $\vec{M}_0(\Omega)\subset\vec{J}_{0,S}(\Omega)$.

Proof. The claimed properties of the operators Q and Q^* follow from Lemma 2.3 and the definitions of generalized and weak solutions of problem (2.77)–(2.79) (as above, we substitute $(0; 0; \psi_3)$ for $\vec{\psi}$). In particular, the property

$$(Q^*\varphi, \vec{\eta})_\Omega = (\varphi, Q\vec{\eta})_0 \quad \forall \varphi \in L_{2,\Gamma} \quad \forall \vec{\eta} \in \vec{L}_2(\Omega) \tag{2.110}$$

follows from the definition

$$E(T\psi_3, \vec{u}) = (A^{1/2}T\psi_3, A^{1/2}\vec{u})_\Omega = (\psi_3, \gamma_n\vec{u})_0, \quad \vec{u} \in \vec{J}^1_{0,S}(\Omega), \tag{2.111}$$

of a generalized solution of problem (2.77)–(2.79) if we substitute $A^{-1/2}\vec{\eta}$ for $\vec{u}$.

Also note that the operator $Q = \gamma_n A^{-1/2} : \vec{J}_{0,S}(\Omega) \to L_{2,\Gamma}$ is compact because $A^{-1/2}$ is a bounded operator from $\vec{J}_{0,S}(\Omega)$ to $\vec{J}^1_{0,S}(\Omega)$, γ_n is a bounded operator from $\vec{J}^1_{0,S}(\Omega)$ to $H^{1/2}_\Gamma$, and $H^{1/2}_\Gamma$ is compactly imbedded in $L_{2,\Gamma}$. □

Consider properties of the operator B defined by relation (2.104). That operator is unbounded and it acts in the space $\vec{J}_{0,S}(\Omega)$. It annihilates elements from $\vec{N}_0(\Omega) \subset \vec{J}_{0,S}(\Omega)$; hence, $\dim \operatorname{Ker} B = \infty$. By virtue of the definition of the operator T, the range of the operator B is a subset of the subspace $\vec{M}_0(\Omega)$. If the operator B_σ has a bounded inverse, then the restriction of the operator B to $\vec{M}_0(\Omega)$ has a bounded inverse because all factors in the definition of B (see (2.104)) have bounded inverse operators. In particular, the restriction of the operator γ_n to $\vec{M}_1(\Omega)$ has a bounded inverse as well (see [16, p. 118]). Then one can consider that

$$\mathcal{D}(B) = \mathcal{R}(B^{-1}) \subset \vec{M}_0(\Omega) \subset \vec{J}_{0,S}(\Omega), \tag{2.112}$$

where $\mathcal{D}(B)$ is a dense set in $\vec{M}_0(\Omega)$.

Then that operator can be extended on $\vec{N}_0(\Omega) = \vec{J}_{0,S}(\Omega) \ominus \vec{M}_0(\Omega)$ as identically zero; therefore, it is defined on a set dense in $\vec{J}_{0,S}(\Omega)$. We denote that set by $\mathcal{D}(B)$ again.

If the equilibrium state is statically stable with respect to the linear approximation, i.e., $B_\sigma \gg 0$ (see (2.85)), then the operator B is nonnegative because the operators Q and Q^* are adjoint to each other due to Lemma 2.4. Then it has a self-adjoint Friedrichs extension; we denote it by B again and denote its domain by $\mathcal{D}(B)$.

Finally, note that the operator B is unbounded and its nonzero eigenvalues $\lambda_k(B)$ have a power-function asymptotic behavior:

$$\lambda_k(B) = \sigma \left(\frac{1}{\pi} \operatorname{mes}\Gamma\right)^{-1/2} k^{1/2}[1 + o(1)] \quad (k \to \infty). \tag{2.113}$$

This is proved in [23, 24].

3. Solvability of the Initial-Boundary Value Problem

Using Cauchy problem (2.98)–(2.100), we investigate the existence and uniqueness of solutions of the original initial-boundary value problem, which is (2.9)–(2.18). In this section, we reduce problem (2.98)–(2.100) to the Cauchy problem for an abstract parabolic equation.

3.1. Passing to the Cauchy problem for a parabolic equation in a Hilbert space. It follows from definition (2.104) of the operator B that the operator $TB_\sigma\gamma_n$ from (2.99) can be represented as follows:

$$B_1 := TB_\sigma\gamma_n = A^{-1/2}BA^{1/2}. \tag{3.1}$$

Since the operator B acts in $\vec{M}_0(\Omega)$ (see (2.112)), while the subspaces $\vec{M}_1(\Omega)$ and $\vec{M}_0(\Omega)$ are linked by relation (2.108), i.e.,

$$\vec{M}_0(\Omega) = A^{1/2}\vec{M}_1(\Omega), \quad \vec{M}_1(\Omega) = A^{-1/2}\vec{M}_0(\Omega), \tag{3.2}$$

it follows that relation (3.1) defines an operator B_1 similar to the operator B and defined on the set $\mathcal{D}(TB_\sigma\gamma_n) \subset \vec{M}_1(\Omega)$ such that

$$\mathcal{R}(B_1) = \mathcal{R}(TB_\sigma\gamma_n) = \vec{M}_1(\Omega). \tag{3.3}$$

This implies that $B_1 = TB_\sigma\gamma_n$ is a positive definite operator acting in $\vec{M}_1(\Omega)$ and such that its general properties and spectrum are the same as for the operator $B : \mathcal{D}(B) \subset \vec{M}_0(\Omega) \to \vec{M}_0(\Omega)$.

Consider the operators

$$R := B^{1/2}PA^{-1/2} : \vec{J}_{0,S}(\Omega) \to \vec{M}_0(\Omega), \quad R^+ := A^{-1/2}PB^{1/2}, \quad \mathcal{D}(R^+) := \mathcal{D}(B^{1/2}) \subset \vec{M}_0(\Omega), \tag{3.4}$$

where $P : \vec{J}_{0,S}(\Omega) \to \vec{M}_0(\Omega)$ is the orthogonal projector on the space $\vec{M}_0(\Omega)$ (one can omit the operator P in (3.4) because $\vec{M}_0(\Omega)$ is invariant for B and $B^{1/2}$ and, therefore, $B^{1/2} = B^{1/2}P = PB^{1/2} = PB^{1/2}P$).

Let us find general properties of the operators R and R^+.

Lemma 3.1. *The following relations are valid for the operators R and R^+:*

$$R \in \mathfrak{S}_\infty\,, \quad R^+ = R^*|\mathcal{D}(B^{1/2}), \quad \overline{R^+} = R^* \in \mathfrak{S}_\infty\,. \tag{3.5}$$

Proof. The second and the third properties in (3.5) follow from (3.4), the definition of an adjoint operator, and the fact that $\mathcal{D}(B^{1/2})$ is dense in $\vec{M}_0(\Omega)$. The first property in (3.5) is proved by Suslina for the case where the domain Ω is piecewise smooth, the boundary-value condition is (2.16), and the functions belong to the domain of the operator B_σ (see (2.82) and Lemma 2.1). If the boundary angle δ satisfies the condition

$$0 < \delta < \delta_* \approx 0,354\pi, \tag{3.6}$$

then the eigenvalues $\lambda_k(R^*R)$ of the operator R^*R have the following power-function asymptotic behavior:

$$\lambda_k(R^*R) = \left(\frac{c}{k}\right)^{1/2}[1 + o(1)], \quad k \to \infty, \quad c = \sigma^2\frac{9}{256\pi}\,\mathrm{mes}\,\Gamma > 0. \tag{3.7}$$

This is proved by Suslina as well (see [25]). □

The operators R^+, R, and R^* are used for formal transformations of problem (2.98)–(2.100), reducing it to the Cauchy problem for an abstract parabolic equation. To do this, we change the variables in (2.98)–(2.100) as follows:

$$\vec{w} = \nu^{-1}R^+\vec{z}. \tag{3.8}$$

Then we apply $\nu\,(R^+)^{-1} = \nu B^{-1/2}PA^{1/2}$ to both sides of (2.99). The operator $\nu\,(R^+)^{-1}$ is a bounded operator from $\vec{M}_1(\Omega)$ to $\mathcal{D}(B^{1/2}) \subset \vec{M}_0(\Omega)$ by virtue of (3.2). This yields the following system of equations:

$$\frac{d\vec{v}}{dt} + \nu^{-1}\frac{d}{dt}(R^+\vec{z}) + \nu A\vec{v} - 2i\omega_0 S_0\vec{v} - 2i\omega_0\nu^{-1}S_0R^+\vec{z} = P_{0,S}\vec{f}, \tag{3.9}$$

$$\frac{d\vec{z}}{dt} + B^{1/2}A^{1/2}(\vec{v} + \nu^{-1}A^{-1/2}B^{1/2}\vec{z}) = \vec{0}. \tag{3.10}$$

It has the following vector-matrix form:

$$\left(\mathcal{I} + \frac{1}{\nu}\mathcal{R}^*\right)\frac{dy}{dt} + (\mathcal{I} + \mathcal{F})\,\mathcal{A}_0 y = f_0(t), \quad y(0) = y^0, \tag{3.11}$$

$$\mathcal{F} := \frac{1}{\nu}\mathcal{R} - 2i\omega_0 S = \frac{1}{\nu}\begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix} - 2i\omega_0\begin{pmatrix} \frac{1}{\nu}S_0A^{-1} & S_0A^{-1/2}B^{-1/2} \\ 0 & 0 \end{pmatrix}, \tag{3.12}$$

$$\mathcal{A}_0 := \begin{pmatrix} \nu A & 0 \\ 0 & \nu^{-1}B \end{pmatrix}, \quad y = \begin{pmatrix} \vec{v} \\ \vec{z} \end{pmatrix}, \quad f_0(t) = \begin{pmatrix} P_{0,S}\vec{f} \\ 0 \end{pmatrix}, \quad y^0 = \begin{pmatrix} \vec{v}(0) \\ \vec{z}(0) \end{pmatrix}. \tag{3.13}$$

Passing from system (3.9), (3.10) to system (3.11)–(3.13), we use the relation

$$\frac{d}{dt}(R^+\vec{z}) = R^*\frac{d\vec{z}}{dt}, \tag{3.14}$$

which is valid due to Lemma 3.1, provided that $\frac{d\vec{z}}{dt}$ is continuous with respect to t.

Lemma 3.2. *The operators* $\mathcal{R}$ *and* $\mathcal{F}$ *from* (3.11) *and* (3.12) *are compact, while the operators* $\mathcal{I} + \nu^{-1}\mathcal{R}^*$ *and* $\mathcal{I} + \mathcal{F}$ *have bounded inverse operators such that*

$$\left(\mathcal{I} + \nu^{-1}\mathcal{R}^*\right)^{-1} = \mathcal{I} - \nu^{-1}\mathcal{R}^*, \quad (\mathcal{I} + \mathcal{F})^{-1} = \mathcal{I} + \mathcal{F}_1, \quad \mathcal{F}_1 \in \mathfrak{S}_\infty. \tag{3.15}$$

Proof. The compactness of $\mathcal{R}$ and $\mathcal{F}$ is obvious because $\mathcal{R} \in \mathfrak{S}_\infty$ by virtue of Lemma 3.1, the operator S_0 is bounded, while A^{-1} and $B^{-1/2}$ are compact. The first property in (3.15) is obvious as well because $\mathcal{R}^*$ is a triangular operator matrix (see (3.12)). Finally, we can check the invertibility of the operator $\mathcal{I} + \mathcal{F}$ by direct computation. Then the structure of the inverse is given by (3.15). □

Apply $\left(\mathcal{I} + \nu^{-1}\mathcal{R}^*\right)^{-1}$, which is a bounded operator with a bounded inverse, to (3.11) from the left. Then we obtain the following Cauchy problem instead of (3.11)–(3.13):

$$\frac{dy}{dt} + (\mathcal{I} + \mathcal{T})\mathcal{A}_0 y = f_0(t), \quad y(0) = y^0, \tag{3.16}$$

$$\mathcal{I} + \mathcal{T} := \left(\mathcal{I} - \nu^{-1}\mathcal{R}^*\right)(\mathcal{I} + \mathcal{F}), \quad \mathcal{T} \in \mathfrak{S}_\infty. \tag{3.17}$$

We took into account that $\left(\mathcal{I} - \nu^{-1}\mathcal{R}^*\right) f_0(t) = f_0(t)$, while the operator $(\mathcal{I} + \mathcal{F})$ has a bounded inverse due to Lemma 3.2.

By virtue of the proven properties of the operators $(\mathcal{I} + \mathcal{F})$ and $\mathcal{A}_0$, problem (3.16)–(3.17) is the Cauchy problem for an abstract parabolic equation. It can be formulated in the form

$$\frac{dy}{dt} = -(\mathcal{I} + \mathcal{T})\mathcal{A}_0 y + f_0(t), \quad y(0) = y^0, \tag{3.18}$$

where the operator $-(\mathcal{I} + \mathcal{T})\mathcal{A}_0$ generates a semigroup which is analytic in a sector containing the positive semiaxis $t > 0$ (see [17, p. 183]). Indeed, the operator $\mathcal{A}_0$ defined by relation (3.13) is self-adjoint and positive definite. Therefore, the operator $-\mathcal{A}_0$ is a generator of an analytic semigroup. The operator $-(\mathcal{I} + \mathcal{T})\mathcal{A}_0$ possesses the same property since $\mathcal{T} \in \mathfrak{S}_\infty$ and $\mathcal{I} + \mathcal{T}$ has a bounded inverse.

3.2. On solvability of the initial-boundary value problem. Using problem (3.18), we investigate the solvability of the original initial-boundary value problem, i.e., (2.9)–(2.18). In the sequel, we say that Cauchy problem (3.18) is associated with problem (2.9)–(2.18).

Definition 3.1. We say that Cauchy problem (3.18) has a *strong solution* $y(t)$ on the segment $[0, T]$ valued in the space $\mathcal{H} := \vec{J}_{0,S}(\Omega) \oplus \vec{M}_0(\Omega)$ if the following conditions are satisfied:

(1) the function $y(t)$ belongs to $\mathcal{D}(\mathcal{A}_0) = \mathcal{D}(A) \oplus \mathcal{D}(B)$ for any $t \in [0, T]$ and $\mathcal{A}_0 y(t) \in C([0, T]; \mathcal{H})$;

(2) we have $\frac{dy}{dt} \in C([0, T]; \mathcal{H})$;

(3) the equation in (3.18) is satisfied for $t \in [0, T]$, while the initial-value condition $y(0) = y^0$ is satisfied for $t = 0$.

Definition 3.2. We say that a function $\varphi(t)$ valued in $\mathcal{H}$ satisfies the *Hölder condition* in a segment $[0, T]$ and denote that as $\varphi(t) \in C^\alpha([0, T]; \mathcal{H})$ if there exist constants $\alpha \in (0, 1]$ and $c_\alpha > 0$ such that

$$\|\varphi(t) - \varphi(s)\|_{\mathcal{H}} \le c_\alpha |t - s|^\alpha, \quad 0 \le s \le t \le T. \tag{3.19}$$

Using [6, Theorem 1.4, p. 130], we obtain the following assertion.

Theorem 3.1. *Let the following conditions be satisfied for problem* (2.9)–(2.18):

$$\vec{f}(t,x) \in C^{\alpha}([0,T];\vec{L}_2(\Omega)), \quad \vec{u}^0 \in \vec{J}_{0,S}(\Omega), \quad \vec{u}^0 = \vec{v}^0 + \vec{w}^0, \tag{3.20}$$

$$\vec{v}^0 \in \mathcal{D}(A) \subset \vec{J}^1_{0,S}(\Omega), \quad \vec{w}^0 \in \vec{M}_1(\Omega) \subset \vec{J}^1_{0,S}(\Omega), \quad \gamma_n \vec{w}^0 \in \mathcal{D}(B_\sigma^{1/2}). \tag{3.21}$$

Then Cauchy problem (3.18) *has a unique strong solution on the segment* $[0,T]$.

Proof. We check the conditions of [6, Theorem 1.4] and consider the transformations above related to the passing from problem (2.9)–(2.18) to problem (3.18).

If the first condition in (3.20) is satisfied, then $P_{0,S}\vec{f} \in C^{\alpha}([0,T];\vec{J}_{0,S}(\Omega))$. Therefore, we have

$$f_0(t) := (P_{0,S}\vec{f};0)^t \in C^{\alpha}([0,T];\mathcal{H}). \tag{3.22}$$

Further, $\vec{v}^0 \in \mathcal{D}(A)$. Thus, to prove that $y^0 = (\vec{v}^0;\vec{z}^0)^t \in \mathcal{D}(\mathcal{A}_0)$ in (3.18), it suffices to check whether $\vec{z}^0 \in \mathcal{D}(B)$. According to definition (3.8), we have

$$B\vec{z}^0 = \nu B(R^+)^{-1}\vec{w}^0 = \nu BB^{-1/2}A^{1/2}\vec{w}^0 = \nu B^{1/2}A^{1/2}\vec{w}^0 \in \vec{M}_0(\Omega). \tag{3.23}$$

Since the last condition in (3.21) is satisfied, we have the inequality

$$\|B_\sigma^{1/2}\gamma_n\vec{w}^0\|_0^2 < \infty.$$

Approximating an element $A^{1/2}\vec{w}^0 \in \vec{M}_0(\Omega)$ by elements $A^{1/2}\vec{w}_k$ from the dense set $\mathcal{D}(B) \subset \mathcal{D}(B^{1/2}) \subset \vec{M}_0(\Omega)$, we have $\big((A^{1/2}T)^* = \gamma_n A^{-1/2}\big)$

$$\begin{aligned}\|B^{1/2}A^{1/2}\vec{w}^0\|_\Omega^2 &= \lim_{k\to\infty}\|B^{1/2}A^{1/2}\vec{w}_k^0\|_\Omega^2 = \lim_{k\to\infty}(BA^{1/2}\vec{w}_k^0, A^{1/2}\vec{w}_k^0)_\Omega \\ = \lim_{k\to\infty}(A^{1/2}TB_\sigma\gamma_n A^{-1/2}A^{1/2}\vec{w}_k^0, A^{1/2}\vec{w}_k^0)_\Omega &= \lim_{k\to\infty}(B_\sigma\gamma_n\vec{w}_k^0, \gamma_n A^{-1/2}A^{1/2}\vec{w}_k^0)_0 \\ &= \lim_{k\to\infty}\|B_\sigma^{1/2}\gamma_n\vec{w}_k^0\|_0^2 = \|B_\sigma^{1/2}\gamma_n\vec{w}^0\|_0^2 < \infty.\end{aligned} \tag{3.24}$$

This implies that (3.23) is satisfied if the last condition in (3.21) is satisfied.

Thus, we proved that condition (3.22) and the condition $y^0 \in \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(A) \oplus \mathcal{D}(B)$ are satisfied. Since the operator $-(\mathcal{I}+\mathcal{T})\mathcal{A}_0$ generates a semigroup analytic in a sector containing the semiaxis $t>0$, the assertion of this theorem follows from [6, Theorem 1.4]. □

Remark 3.1. To deduce (3.24), we used the relation

$$\|B^{1/2}\vec{\varphi}\|_\Omega^2 = \|B_\sigma^{1/2}\gamma_n A^{-1/2}\vec{\varphi}\|_0^2, \tag{3.25}$$

which is checked (in the same way as with (3.24)) for elements of $\mathcal{D}(B)$; if we pass to the closure, then (3.25) is valid for elements of $\mathcal{D}(B^{1/2})$.

Remark 3.2. It follows from Theorem 3.1 that conditions (3.20) and (3.21) are sufficient for the existence of a strong solution of problem (3.11)–(3.13) on the segment $[0,T]$. Then they are sufficient for the existence of a strong solution of Cauchy problem (3.9)–(3.10) on the same segment such that all terms of (3.9) belong to $C([0,T];\vec{J}_{0,S}(\Omega))$, all terms of (3.10) belong to $C([0,T];\vec{M}_0(\Omega))$, and

$$B^{1/2}A^{1/2}(\vec{v} + \nu^{-1}A^{-1/2}B^{1/2}\vec{z}) = B^{1/2}A^{1/2}\vec{v} + \nu^{-1}B\vec{z}. \tag{3.26}$$

Indeed, if conditions (3.20) and (3.21) are satisfied, then problem (3.16)–(3.17) has a strong solution on $[0,T]$. Therefore, due to the first relation in (3.15), problem (3.11)–(3.13) has a strong solution on $[0,T]$. Finally, using relation (3.14), which holds in the considered case, we conclude that Eqs. (3.9) and (3.10), which follow from Eqs. (3.11)–(3.13), are satisfied.

Definition 3.3. We say that a solution of associated Cauchy problem (3.18) possesses additional smoothness properties with respect to the variable $t \in [0,T]$ if the following conditions are satisfied:

$$B\vec{z}(t) \in C([0,T];\mathcal{D}(B^{1/2})), \quad PA\vec{v}(t) \in C([0,T];\mathcal{D}(B^{1/2})). \tag{3.27}$$

Note that the conditions

$$\vec{z}^0 \in \mathcal{D}(B^{3/2}) \subset \vec{M}_0(\Omega), \quad PA\vec{v}^0 \in \mathcal{D}(B^{1/2}) \subset \vec{M}_0(\Omega) \tag{3.28}$$

are necessary for conditions (3.27).

Theorem 3.2. *Let a solution of problem* (3.18) *possess additional smoothness properties with respect to* t*. Let conditions* (3.28) *and the first condition in* (3.20) *be satisfied. Then problem* (3.9)–(3.10) *has a unique strong solution on the segment* $[0,T]$ *such that any term in* (3.10) *belongs to* $C([0,T];\mathcal{D}(B^{1/2}))$ *for that strong solution and problem* (2.98)–(2.100) *has a unique strong solution such that any term in* (2.98) *belongs to* $C([0,T];\vec{J}_{0,S}(\Omega))$ *and any term in* (2.99) *belongs to* $C([0,T];\vec{J}^1_{0,S}(\Omega))$ *for that strong solution.*

Proof. If the conditions of the theorem are satisfied, then problem (3.18) has a unique strong solution with additional smoothness properties with respect to t. Then, by virtue of Remark 3.2, problem (3.9)–(3.10) has a unique strong solution on $[0,T]$ and property (3.26) holds. Hence, by virtue of (3.27), any term in (3.10) belongs to $C([0,T];\mathcal{D}(B^{1/2}))$.

Applying the bounded operator $\nu^{-1}R^* = \nu^{-1}\overline{R^+}$, $R^+ = A^{-1/2}PB^{1/2} = R^*\big|\mathcal{D}(B^{1/2})$, to both sides of (3.10), we obtain the relation

$$\frac{d}{dt}\left(\nu^{-1}R^+\vec{z}\right) + \nu^{-1}TB_\sigma\gamma_n(\vec{v} + \nu^{-1}R^+\vec{z}) = \vec{0} \tag{3.29}$$

(relations (3.14) and (3.1) are used to deduce it). Any term on the left-hand side of (3.29) belongs to $C([0,T];\vec{M}_1(\Omega))$. Therefore, using substitution (3.8) in (3.9) and (3.29), we conclude that problem (2.98)–(2.100) has a unique strong solution on $[0,T]$ such that any term in (2.98) belongs to $C([0,T];\vec{J}_{0,S}(\Omega))$ and any term in (2.99) belongs to $C([0,T];\vec{M}_1(\Omega)) \subset C([0,T];\vec{J}^1_{0,S}(\Omega))$ for that strong solution. □

Corollary 3.1. *Let the conditions of Theorem* 3.2 *be satisfied. Let the following conditions be satisfied*:

$$B_\sigma\zeta^0 \in (H^{1/2}_\Gamma)^*, \quad \nu\vec{w}^0 + TB_\sigma\zeta^0 = 0. \tag{3.30}$$

Then problem (2.93)–(2.94) *has a unique strong solution on* $[0,T]$ *such that*

$$B_\sigma\zeta(t) := B_\sigma\left(\zeta^0 + \int_0^t (\gamma_n\vec{u})(s)ds\right) \in C^1([0,T];(H^{1/2}_\Gamma)^*) \tag{3.31}$$

and

$$\vec{w}(t) \in C([0,T];\vec{M}_1(\Omega)). \tag{3.32}$$

Proof. By virtue of Theorem 3.2, we have

$$TB_\sigma\gamma_n(\vec{v} + \vec{w}) = TB_\sigma\gamma_n\vec{u} = TB_\sigma\frac{d\zeta}{dt} \in C([0,T];\vec{M}_1(\Omega)). \tag{3.33}$$

Therefore, we have

$$\vec{w}(t) = -\frac{1}{\nu}\int_0^t TB_\sigma\gamma_n\vec{u}(s)ds + \vec{w}^0 \in C^1([0,T];\vec{M}_1(\Omega)),$$

i.e., property (3.32) holds. It follows from Sec. 2 that property (3.33) holds if and only if

$$B_\sigma\gamma_n u = B_\sigma\frac{d\zeta}{dt} \in C([0,T];(H^{1/2}_\Gamma)^*). \tag{3.34}$$

This and conditions (3.30) imply that property (3.31) and the the first equation in (2.94) are satisfied. □

Remark 3.3. Conditions (3.30) are satisfied if

$$\zeta^0 \in \mathcal{D}(B_\sigma) \subset L_{2,\Gamma}, \quad \nu(\partial\vec{w}^0)_3 + B_\sigma\zeta^0 = 0, \quad (\partial\vec{w}^0)_1 = (\partial\vec{w}^0)_2 = 0. \tag{3.35}$$

To conclude this section, we note that Theorems 3.1 and 3.2 and Corollary 3.1 allow us to find sufficient conditions of strong solvability for original initial-boundary value problem (2.9)–(2.18) (i.e., for problem (2.93)–(2.94) with relation (2.89)), provided that solutions of Cauchy problem (3.18) possess an additional smoothness (see Definition 3.3).

4. Normal Oscillations of the System under the Condition of Static Stability with Respect to the Linear Approximation

In this section, we study the problem of normal oscillations for a capillary viscous fluid; spectral properties and the basisness of eigenelements and associated elements are considered.

4.1. Spectral problem: the posing and elementary properties of solutions. Consider solutions of homogeneous problem (2.102)–(2.104) or transformed problem (3.11)–(3.13), depending on t as $\exp(-\lambda t)$. Such solutions are called *normal oscillations* of the hydraulic system.

For the corresponding amplitude elements, problem (2.102)–(2.104) leads to the spectral problem

$$\nu A\vec{\xi} - 2i\omega_0 A^{1/2} S_0 A^{-1/2}(\vec{\xi}+\vec{\eta}) - \nu^{-1}B(\vec{\xi}+\vec{\eta}) = \lambda\vec{\xi}, \quad \vec{\xi} \in \mathcal{D}(A), \tag{4.1}$$

$$\nu^{-1}B(\vec{\xi}+\vec{\eta}) = \lambda\vec{\eta}, \quad P(\vec{\xi}+\vec{\eta}) \in \mathcal{D}(B), \tag{4.2}$$

where $P : \vec{J}_{0,S}(\Omega) \to \vec{M}_0(\Omega)$ is an orthogonal projector.

In the same way, problem (3.9), (3.10) leads to a spectral problem represented in the following vector-matrix form (cf. problem (3.11), (3.12)):

$$(\mathcal{I}+\mathcal{F})\,\mathcal{A}_0 y = \lambda\left(\mathcal{I} + \frac{1}{\nu}\mathcal{R}^+\right)y, \quad y = (\vec{v};\vec{z})^t \in \mathcal{D}(\mathcal{A}_0). \tag{4.3}$$

Similarly, problem (3.11), (3.12) leads to the problem

$$(\mathcal{I}+\mathcal{F})\,\mathcal{A}_0 y = \lambda\left(\mathcal{I} + \frac{1}{\nu}\mathcal{R}^*\right)y, \quad y \in \mathcal{D}(\mathcal{A}_0), \tag{4.4}$$

$$\mathcal{R}^+ = \begin{pmatrix} 0 & R^+ \\ 0 & 0 \end{pmatrix}, \quad \mathcal{R}^* = \begin{pmatrix} 0 & R^* \\ 0 & 0 \end{pmatrix}. \tag{4.5}$$

Consider elementary properties of solutions of problems (4.1)–(4.2), (4.3), and (4.4).

(1) The number $\lambda = 0$ is an eigenvalue for none of those problems.
Indeed, for $\lambda = 0$, we sequentially have $(\mathcal{I}+\mathcal{F})\mathcal{A}_0 y = 0$, $\mathcal{A}_0 y = 0$, and $y = 0$ in problem (4.3) because $\mathcal{I}+\mathcal{F}$ and $\mathcal{A}_0$ are invertible (see Lemma 3.2 and relation (3.13) and take into account that $A \gg 0$ and $B \gg 0$).

(2) All eigenvalues λ are located in the right-hand half-plane.
Indeed, if $\lambda \neq 0$, then, from problem (4.1), (4.2), we pass to the system

$$\nu\vec{\xi} = 2i\omega_0 A^{-1/2} S_0 A^{-1/2}(\vec{\xi}+\vec{\eta}) + \lambda A^{-1}(\vec{\xi}+\vec{\eta}), \quad \nu\vec{\eta} = \lambda^{-1}B(\vec{\xi}+\vec{\eta}). \tag{4.6}$$

From the above system, we pass to the equation

$$(\nu I - 2i\omega_0 S)\vec{\delta} = \lambda A^{-1}\vec{\delta} + \lambda^{-1}B\vec{\delta}, \quad \vec{\delta} := \vec{\xi}+\vec{\eta}, \quad S := A^{-1/2}S_0A^{-1/2}. \tag{4.7}$$

This yields

$$\operatorname{Re}\lambda = \frac{\nu\|\vec{\delta}\|^2_\Omega}{(\|A^{-1/2}\vec{\delta}\|^2_\Omega + |\lambda|^{-2}\|B^{1/2}\vec{\delta}\|^2_\Omega)} > 0 \tag{4.8}$$

(we take into account that $S = S^*$).

(3) Problems (4.3) (with the operator $\mathcal{R}^+$) and (4.4) (with the operator $\mathcal{R}^*$) have the same solutions $y \in \mathcal{D}(\mathcal{A}_0)$.
Indeed, if $y = (\vec{v}; \vec{z})^t \in \mathcal{D}(\mathcal{A}_0)$, then $\vec{z} \in \mathcal{D}(B)$; then

$$R^+\vec{z} = R^*\vec{z}, \quad \mathcal{R}^+ y = \mathcal{R}^* y \tag{4.9}$$

due to Lemma 3.1 and relations (4.5).

(4) Only a discrete spectrum, i.e., a spectrum consisting of isolated eigenvalues of finite multiplicities with the limit point $\lambda = \infty$, is possible for the specified problems.
Indeed, problem (4.4) is equivalent to the spectral problem

$$y = \lambda \mathcal{A}_0^{-1} (\mathcal{I} + \mathcal{F})^{-1} \left(\mathcal{I} + \frac{1}{\nu} \mathcal{R}^* \right) y, \quad y \in \mathcal{H}, \tag{4.10}$$

on characteristic numbers of the compact operator

$$\mathcal{A}_0^{-1} (\mathcal{I} + \mathcal{F})^{-1} \left(\mathcal{I} + \frac{1}{\nu} \mathcal{R}^* \right) = \mathcal{A}_0^{-1} (\mathcal{I} + \mathcal{T}), \quad \mathcal{T} \in \mathfrak{S}_\infty, \tag{4.11}$$

which is a weak perturbation of the self-adjoint operator $\mathcal{A}_0$.

Using the latter property, we prove the basisness property for the system of eigenelements and associated elements and find the asymptotic behavior of eigenvalues for problem (4.11). The necessary definitions are provided below.

4.2. On the Abel–Lidskii basisness for the system of root elements. First of all, we note that this property is intermediate between the basisness and the basisness with brackets: any element is expanded into a series by means of a special method called the Abel–Lidskii method of order α (see [1, p. 248–249]).

Let an operator L have a discrete countable spectrum consisting of eigenvalues of finite multiplicities $\{\mu_j\}_{j=1}^\infty$ (characteristic numbers of the operator $A = L^{-1}$) located, maybe, apart from a finite number of them, in the sector

$$\Lambda_\theta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\}. \tag{4.12}$$

Also, let $\alpha > 0$, $\alpha\theta < \pi/2$, $\lambda^\alpha := |\lambda|^\alpha e^{i\alpha \arg \lambda}$, $\{f_j\}_{j=1}^\infty$ be the system of root elements (i.e., eigenelements and associated elements) of the operator L corresponding to the eigenvalues $\{\mu_j\}_{j=1}^\infty$, and $\{g_j\}_{j=1}^\infty$ be a system biorthogonal to the system $\{f_j\}_{j=1}^\infty$.

First, we assume that the system $\{f_j\}_{j=1}^\infty$ does not contain associated elements of the operator L, corresponding to the eigenvalues $\mu_j \in \Lambda_\theta$.

Definition 4.1. We say that $\{f_j\}_{j=1}^\infty$ is an *Abel–Lidskii system of order α of elements* if there exists an increasing sequence of indices $0 = m_0 < m_1 < \ldots < m_l < \ldots$ such that the series

$$\sum_{l=0}^{\infty} P_l(t) f, \quad P_l(t) f := \sum_{j=m_l+1}^{m_{l+1}} (f, g_j) e_j(t) f_j, \tag{4.13}$$

$$e_j(t) := e_j(t; \alpha) := \exp(-\mu_j^\alpha t) \quad (\mu_j \in \Lambda_\theta), \quad e_j(t) \equiv 1 \quad (\mu_j \overline{\in} \Lambda_\theta), \tag{4.14}$$

converges for positive t and its sum $f(t)$ tends to f as $t \to +0$.

Definition 4.1 can be extended for the general case where associated elements of the operator L exist. Let $f_p, \ldots, f_q$ be the basis of the subspace of root elements of the operator L corresponding to the eigenvalue $\mu \in \Lambda_\theta$. Then the sum $\sum\limits_{j=p}^{q} (f, g_j) e_j(t) f_j$ from Definition (4.13) is replaced by the integral

$$-\frac{1}{2\pi i} \oint\limits_{|\lambda - \mu| = \varepsilon} \exp(-\lambda^\alpha t) R_L(\lambda) f \, d\lambda, \tag{4.15}$$

where the path of integration is located in Λ_θ and encircles the isolated eigenvalue μ of the operator L, while $R_L(\lambda) := (L - \lambda I)^{-1}$. If $t = 0$, then the integral (4.15) is equal to the projection of the element f on the subspace of root elements of the operator L corresponding to the eigenvalue μ. If μ is a simple pole of the resolvent $R_L(\lambda)$, then that integral is equal to $\sum_{j=p}^{q}(f, g_j)e_j(t)f_j$.

If a compact operator A has no unbounded inverse operator L, i.e., if $\operatorname{Ker} A \neq \{0\}$, then, in (4.15), we take the modified resolvent $A(I-\lambda A)^{-1}$ instead of $R_L(\lambda)$ and characteristic numbers of the operator A instead of μ_j.

Assume that the operator A, i.e., the operator inverse to the operator L with discrete spectrum $\{\mu_j\}_{j=1}^{\infty}$, is represented as

$$A = A_0(I + T_1), \tag{4.16}$$

where T_1 is a compact operator, while A_0 is a self-adjoint compact operator of class $\mathcal{S}^{(p)} \subset \mathfrak{S}_p$, i.e., its s-numbers (eigenvalues of the operator $(A_0^2)^{1/2}$) satisfy the estimate

$$s_j(A_0) \le c(A_0)j^{-p}, \quad j = 1, 2, \dots. \tag{4.17}$$

Also, assume that zero is not an eigenvalue of the operator A_0 and all eigenvalues of the operator A_0 are positive (negative), maybe, apart from a finite number of them. Further, we need the following assertion.

Theorem 4.1 (see [1, p. 292])**.** *If the above assumptions are satisfied, then*:

(1) *the system of root elements* $\{f_j\}_{j=1}^{\infty}$ *of the operator* A *from* (4.16) *is an Abel–Lidskii basis of order* $\alpha > p^{-1}$;

(2) *if characteristic numbers of the operator* A_0 *have the asymptotic behavior*

$$\mu_j(A_0) = c_{A_0}j^p + o(j^p), \quad j \to \infty, \quad c_{A_0} \neq 0, \tag{4.18}$$

then characteristic numbers $\mu_j(A)$ *of the operator* A *have the same asymptotic behavior.*

4.3. On the basisness of modes of normal oscillations and asymptotic behavior of the spectrum. Note that normal oscillations are studied under condition (2.85), which is the condition of static stability with respect to the linear approximation. Then, as we know from Sec. 4.1, problems (4.1)–(4.2), (4.3), and (4.4) are equivalent to each other.

Theorem 4.2. *The following assertions are valid*:

(1) *The spectrum of problem* (4.10) *is discrete and its limit point is* $\lambda = \infty$*. Thus, the same holds for problems* (4.1)–(4.2), (4.3), *and* (4.4).

(2) *All eigenvalues* λ *have finite multiplicities and are located* (*maybe, apart from a finite number of them*) *in the sector* Λ_ε *for any positive* ε.

(3) *Root elements* (*eigenelements and associated elements*) *of problem* (4.10) *form an Abel–Lidskii basis of order* $\alpha > 2$.

(4) *Asymptotic behavior of eigenvalues* λ_j *of problem* (4.10) *is as follows*:

$$\lambda_j = \sigma\nu^{-1}\left(\frac{1}{\pi}\operatorname{mes}\Gamma\right)^{-1/2} j^{1/2}[1 + o(1)] \quad (j \to \infty). \tag{4.19}$$

Proof. 1. First, we note that the operator $\mathcal{A}_0^{-1} = \operatorname{diag}(\nu^{-1}A^{-1}; \nu B^{-1})$ (see (3.13)) acting in the space $\mathcal{H} = \vec{J}_{0,S}(\Omega) \oplus \vec{M}_0(\Omega)$ is a compact positive operator and the eigenvalues $\lambda_j(A)$ (characteristic numbers of the operator A^{-1}) have the asymptotic behavior

$$\lambda_j(A) = \left(\frac{\operatorname{mes}\Omega}{3\pi^2}\right)^{-2/3} j^{2/3}[1 + o(1)], \quad j \to \infty, \tag{4.20}$$

(see [18] and [16, p. 279]), while the eigenvalues $\lambda_j(B)$ have the asymptotic behavior (2.113).

Let

$$n(r, G) := \sum_{\mu_j(G) \le r} 1 \tag{4.21}$$

denote the distribution function for characteristic numbers of the compact positive operator G. Then it follows from (4.20) and (2.113) that

$$\lim_{r\to\infty} n(r, A^{-1})r^{-3/2} = \frac{\text{mes}\,\Omega}{3\pi^2}, \quad \lim_{r\to\infty} n(r, B^{-1})r^{-2} = \sigma^{-2}\left(\frac{1}{\pi}\,\text{mes}\,\Gamma\right). \tag{4.22}$$

Since

$$n(r, \mathcal{A}_0^{-1}) = n(r, \nu^{-1}A^{-1}) + n(r, \nu B^{-1}), \tag{4.23}$$

it follows from (4.22) that

$$\lim_{r\to\infty} n(r, \mathcal{A}_0^{-1})r^{-2} = \lim_{r\to\infty} n(r, \nu B^{-1})r^{-2} = \nu^2\sigma^{-2}\left(\frac{1}{\pi}\,\text{mes}\,\Gamma\right) > 0. \tag{4.24}$$

This yields the asymptotic relation

$$\lambda_j(\mathcal{A}_0^{-1}) = \lambda_j(\nu B^{-1})[1 + o(1)] = \nu\sigma^{-1}\left(\frac{1}{\pi}\,\text{mes}\,\Gamma\right)^{1/2} j^{-1/2}[1 + o(1)]. \tag{4.25}$$

2. It follows from (4.25) that the operator $\mathcal{A}_0^{-1}$ belongs to $\mathcal{S}^{(p)}$ for $p = 1/2$. Since $\mathcal{F} \in \mathfrak{S}_\infty$ in problem (4.10)–(4.11), it follows from the first assertion of Theorem 4.1 that root elements of problem (4.10), i.e., the eigenvalue problem for the operator (4.11), form an Abel–Lidskii basis of order $\alpha > p^{-1} = 2$.

3. It follows from (4.25) and the second assertion of Theorem 4.1 that problem (4.10) has a discrete spectrum $\{\lambda_j\}_{j=1}^\infty$ such that the eigenvalues satisfy asymptotic relation (4.19).

4. The fact that the spectrum is localized in the sector Λ_ε for any positive ε follows from the Keldysh theorem (see, e.g., [5, p. 314–317]).

This completes the proof of Theorem 4.2. □

Remark 4.1. The proof of Theorem 4.2 for the case where the vessel does not rotate is given in the diploma thesis of Dudik composed under the guidance of the author of the present paper (see [3]).

Thus, the final conclusion for spectral problems (4.1)–(4.2), (4.3), and (4.4) is as follows: if the condition of static stability with respect to the linear approximation is satisfied, then all normal motions of a capillary viscous fluid in a uniformly rotating vessel are asymptotically stable, while the spectrum and the root functions possess the properties proved above.

5. The Inversion of Lagrange's Theorem on Stability

In this section, we provide a detailed proof of the inversion of Lagrange's theorem on stability for small motions of a capillary fluid. We begin from the case where the fluid does not rotate. Then we come to the case where the fluid uniformly rotates under the assumption of relative equilibrium. Note that the specified problem is briefly considered in [8–10, 28]; in [13, 14, Vol. 2], it is treated in more detail, but still incompletely.

5.1. On unstable motions for non-rotating fluid. Consider spectral problem (4.1)–(4.2) for $\omega_0 = 0$, assuming that the operator B acts from $\mathcal{D}(B) \subset \vec{M}_0(\Omega)$ to $\vec{M}_0(\Omega)$. Then, from (4.1)–(4.2), we obtain the problem

$$A\vec{\xi} - \alpha B(P\vec{\xi} + \vec{\eta}) = \widetilde{\lambda}\vec{\xi}, \quad \vec{\xi} \in \mathcal{D}(A), \quad \alpha = \nu^{-2}, \quad \widetilde{\lambda} = \lambda/\nu, \tag{5.1}$$

$$\alpha B(P\vec{\xi} + \vec{\eta}) = \widetilde{\lambda}\vec{\eta}, \quad (P\vec{\xi} + \vec{\eta}) \in \mathcal{D}(B), \tag{5.2}$$

where the operator P is the orthogonal projector from $\vec{J}_{0,S}(\Omega)$ on $\vec{M}_0(\Omega)$.

We recall (see (2.104)) that $B = Q^* B_\sigma Q$, $Q = \gamma_n A^{-1/2}$, and $Q^* = A^{1/2} T$, where B_σ is the operator of potential energy (2.82) assumed to be positive definite in $L_{2,\Gamma}$. Now we assume that the infimum of the operator B_σ is negative, i.e., condition (2.85) (of the static stability with respect to the linear approximation) is not satisfied. Since the operator B_σ has a discrete real spectrum $\{\lambda_k(B_\sigma)\}_{k=1}^{\infty}$ with the limit point $\lambda = +\infty$ (see Lemma 2.1), in the sequel, we consider the general case where the operator B_σ has κ negative eigenvalues (counted with their multiplicities) and zero is the eigenvalue of multiplicity q:

$$\lambda_1(B_\sigma) \leq \ldots \leq \lambda_\kappa(B_\sigma) < 0 = \lambda_{\kappa+1}(B_\sigma) = \ldots = \lambda_{\kappa+q}(B_\sigma) < \lambda_{\kappa+q+1}(B_\sigma) \leq \ldots . \tag{5.3}$$

Note that eigenvalues of the operator B_σ (and other characteristics of operators of problem (5.1)–(5.2)) are continuous functions of parameters of the considered hydraulic system. In particular, if the intensity of the gravitational field changes its sign and absolute value, then the least eigenvalue of the operator B_σ coinciding with its infimum might become negative; then condition (5.3) with $\kappa \geq 1$ and $q \geq 0$ is satisfied.

Theorem 5.1. *If conditions* (5.3) *are satisfied, then the operator* $B\colon \mathcal{D}(B) \subset \vec{M}_0(\Omega) \to \vec{M}_0(\Omega)$ *has a discrete real spectrum and its eigenvalues* $\lambda_k(B)$, $k = 1, 2, \ldots$, *satisfy inequalities of kind* (5.3):

$$\lambda_1(B) \leq \ldots \leq \lambda_\kappa(B) < 0 = \lambda_{\kappa+1}(B) = \ldots = \lambda_{\kappa+q}(B) < \lambda_{\kappa+q+1}(B) \leq \ldots . \tag{5.4}$$

Proof. Consider the spectral problem

$$B\vec{\eta} := Q^* B_\sigma Q \vec{\eta} = \lambda \vec{\eta}, \quad \vec{\eta} \in \mathcal{D}(B) \subset \vec{M}_0(\Omega). \tag{5.5}$$

Since the operator B is assumed to act in the space $\vec{M}_0(\Omega)$, it follows that the operator Q has an inverse (like the operator Q^*); this fact directly follows from Lemmas 2.3 and 2.4. Therefore, we can change the variables in (5.5):

$$Q\vec{\eta} = \gamma_n A^{-1/2} \vec{\eta} =: \zeta \in \mathcal{D}(B_\sigma). \tag{5.6}$$

Then, applying the operator Q from the left, we obtain the problem

$$C B_\sigma \zeta = \lambda \zeta, \quad \zeta \in \mathcal{D}(B_\sigma) \subset L_{2,\Gamma}, \quad C := \gamma_n T, \tag{5.7}$$

where the operator C is such that

$$0 < C \in \mathfrak{S}_\infty(L_{2,\Gamma}). \tag{5.8}$$

Indeed, T is a bounded operator from $(H_\Gamma^{1/2})^* \supset L_{2,\Gamma}$ to $\vec{M}_1(\Omega)$, while γ_n is a bounded operator from $\vec{M}_1(\Omega)$ to $H_\Gamma^{1/2} \subset L_{2,\Gamma}$. Since the imbeddings $L_{2,\Gamma} \subset (H_\Gamma^{1/2})^*$ and $H_\Gamma^{1/2} \subset L_{2,\Gamma}$ are compact, it follows that the operator $C = \gamma_n T : L_{2,\Gamma} \to L_{2,\Gamma}$ is compact.

To prove that the operator C is positive, we apply abstract identity (2.48) to the spaces $F = \vec{J}_{0,S}^1(\Omega)$ and $G = L_{2,\Gamma}$ and the corresponding operators $T_M = T$ and $\gamma_M = \gamma_n$. Then, for $\vec{\eta} \in \vec{M}_1(\Omega)$ and $\psi \in L_{2,\Gamma}$, we have

$$(\vec{\eta}, T\psi)_{1,\Omega} = \langle \gamma_n \vec{\eta}, \psi \rangle_0 = (\gamma_n \vec{\eta}, \psi)_0. \tag{5.9}$$

This implies that $\gamma_n = T^*$. Therefore, $C = \gamma_n T = C^* \geq 0$. Since the operators γ_n and T are invertible on $\vec{M}_1(\Omega)$, it follows that $C > 0$.

Assigning $\vec{\eta} = T\psi$ in (5.9), we see that

$$\|T\vec{\psi}\|_{1,\Omega}^2 = (C\psi, \psi)_0 = \|C^{1/2}\psi\|_0^2. \tag{5.10}$$

If $\varphi = C\psi$ and $\psi \in L_{2,\Gamma}$, then (5.10) implies the identity

$$\|T\psi\|_{1,\Omega}^2 = (\varphi, C^{-1}\varphi)_0 = \|C^{-1/2}\varphi\|_0^2 < \infty. \tag{5.11}$$

This yields

$$\mathcal{D}(C^{-1/2}) = H_\Gamma^{1/2} \tag{5.12}$$

because

$$\varphi = \gamma_n(T\psi) = \gamma_n \vec{w}, \quad \vec{w} \in \vec{M}_1(\Omega) \subset \vec{J}_{0,S}^1(\Omega). \tag{5.13}$$

Hence, we have

$$\varphi = \gamma_n \vec{w} \in H_\Gamma^{1/2}. \tag{5.14}$$

For problem (5.7), this means that any solution ζ from $\mathcal{D}(B_\sigma)$ belongs to $\mathcal{D}(C^{-1})$ as well provided that $\lambda \neq 0$. Therefore, problem (5.7) is equivalent to the problem

$$B_\sigma \zeta = \lambda C^{-1}\zeta, \quad \zeta \in \mathcal{D}(B_\sigma) \cap \mathcal{D}(C^{-1}\zeta), \tag{5.15}$$

where the operator C^{-1} is self-adjoint, positive definite, and unbounded. Obviously, the spectrum of problem (5.15) coincides with the spectrum of problem (5.5). Let us prove that problem (5.15) has a discrete spectrum and its eigenvalues possess properties (5.4).

Instead of problem (5.15), consider the problem

$$B_{\sigma,b}\zeta = \lambda C^{-1}\zeta, \quad B_{\sigma,b} := B_\sigma + bI, \quad \mathcal{D}(B_{\sigma,b}) = \mathcal{D}(B_\sigma), \tag{5.16}$$

where a positive constant b is such that the operator $B_{\sigma,b}$ is positive definite:

$$B_{\sigma,b} \gg 0. \tag{5.17}$$

Then, as we see from the above transformation, problem (5.16) is equivalent to the spectral problem

$$B_b\vec{\eta} := Q^*(B_\sigma + bI)Q\vec{\eta} = \lambda\vec{\eta}, \tag{5.18}$$

which is already studied above (see Lemmas 2.1 and 2.4 and the end of Sec. 2.4). In particular, problem (5.18) has a discrete positive spectrum with the limit point $\lambda = +\infty$, while its eigenvalues have an asymptotic behavior defined by (2.113).

Using the operator $B_{\sigma,b}$, we can represent problem (5.15) as follows:

$$(B_{\sigma,b} - bI)\zeta = \lambda C^{-1}\zeta. \tag{5.19}$$

Performing the transformations inverse to the ones above and coming back to problem (5.5), we represent the latter one as follows:

$$(B_b - b\widetilde{C})\vec{\eta} = \lambda\vec{\eta}, \quad B_b = Q^* B_{\sigma,b} Q, \quad \widetilde{C} := Q^*Q = (A^{1/2}T)(\gamma_n A^{-1/2}). \tag{5.20}$$

It follows from the above that the operator $\widetilde{C}$ is compact and positive (as the operator $C = \gamma_n T = (\gamma_n A^{-1/2})(A^{1/2}T) = QQ^*$) and its spectrum coincides with the spectrum of the operator $C = \gamma_n T = (\gamma_n A^{-1/2})(A^{1/2}T) = QQ^*$. Since B_b is positive definite, we can change the variables in (5.20):

$$B_b^{1/2}\vec{\eta} = \vec{v}. \tag{5.21}$$

This yields the following spectral problem instead of (5.20):

$$(I - bB_b^{-1/2}\widetilde{C}B_b^{-1/2} - \lambda B_b^{-1})\vec{\eta} = 0, \quad \vec{\eta} \in \vec{M}_0(\Omega), \tag{5.22}$$

where B_b^{-1} and $B_b^{-1/2}CB_b^{-1/2}$ are compact positive operators such that eigenvalues of B_b^{-1} have a power asymptotic behavior implied from (2.113). This and [5, Theorem 11.1] imply that problem (5.22) has a discrete real spectrum with the limit point $\lambda = +\infty$, while its eigenvalues have asymptotic behavior (2.113).

Thus, problem (5.5) equivalent to problem (5.22) has a discrete real spectrum with the specified asymptotic behavior of eigenvalues. This means that the operator B in (5.5) is bounded from below and self-adjoint while its spectrum is discrete. Then its eigenvalues λ_k, $k = 1, 2, \ldots$, are consecutive minima of the variational ratio $(B\vec{\eta}, \vec{\eta})/(\vec{\eta}, \vec{\eta})$, which can be represented as

$$\frac{\displaystyle\int_\Gamma [|\nabla_\Gamma(\gamma_n\vec{w})|^2 + a_\Gamma|\gamma_n\vec{w}|^2]\, d\Gamma}{E(\vec{w}, \vec{w})}, \quad \vec{w} = A^{-1/2}\vec{\eta} \in \vec{M}_1(\Omega), \quad \gamma_n\vec{w} \in H_0^1(\Gamma) \cap L_{2,\Gamma}, \tag{5.23}$$

(see (2.83)).

Since the denominator of (5.23) is positive for $\vec{w} \neq \vec{0}$, it follows that the sign of an eigenvalue λ is determined by the sign of the quadratic form at the numerator. Therefore, computing consecutive (according to their decrease) minima of variational fracture (5.23) and taking into account (5.3), we see that the sign of the quadratic form at the numerator is negative at a subspace of dimension κ. This implies that the number of negative eigenvalues of problem (5.5) is equal to κ. Further, since that quadratic form takes zero value at a subspace of dimension q, it follows that the q eigenvalues of problem (5.5), following the negative ones, are equal to zero. Finally, the eigenvalues $\lambda_{\kappa+q+1}(B)$ are negative for $j \geq 1$ and tend to $+\infty$ as $j \to \infty$.

Thus, we proved that inequalities (5.4) are satisfied for eigenvalues of the operator B if inequalities (5.3) hold. □

Consider properties of solutions of system (5.1)–(5.2), assuming that conditions (5.3) are satisfied; then conditions (5.4) are satisfied as well by virtue of Theorem 5.1. For simplicity, we denote $\widetilde{\lambda}$ from (5.1)–(5.2) by λ.

Lemma 5.1. *If* Ker $B \neq \{0\}$ *and* $q > 0$ *in inequalities* (5.4), *then problem* (5.1)–(5.2) *has a solution*

$$\lambda = \lambda_0 = 0, \quad \vec{\eta} = \vec{\eta}_0 = \vec{\psi}, \quad \forall \vec{\psi} \in \text{Ker } B, \quad \vec{\xi} = \vec{0}, \tag{5.24}$$

called a transitional (*from the right-hand half-plane to the left-hand one*) *solution of problem* (5.1)–(5.2).

Proof. Assigning $\lambda = 0$ in (5.1)–(5.2), we have

$$A\vec{\xi} - \alpha B(P\vec{\xi} + \vec{\eta}) = \vec{0}, \quad \alpha B(P\vec{\xi} + \vec{\eta}) = 0.$$

This implies that $A\vec{\xi} = \vec{0}$ (therefore, $\vec{\xi} = 0$); then $B\vec{\eta} = \vec{0}$ (therefore, $\vec{0} \neq \vec{\eta} \in \text{Ker } B$). □

The further investigation of properties of solutions for problem (5.1)–(5.2) is related to the proof of the so-called stability change principle.

Denote the kernel of the operator B by $\vec{H}_0$: $\vec{H}_0 = \text{Ker } B \neq \{0\}$, $\dim \vec{H}_0 = q > 0$. Denote its orthogonal complement in $\vec{M}_0(\Omega)$ by $\vec{H}_1$: $\vec{H}_1 := \vec{M}_0(\Omega) \ominus \vec{H}_0$. Then problem (5.1)–(5.2) takes the form

$$A\vec{\xi} - \alpha B_1(\vec{\xi} + \vec{\eta}) = \lambda\vec{\xi}, \quad \vec{\xi} \in \mathcal{D}(A), \tag{5.25}$$

$$\alpha B_1(\vec{\xi} + \vec{\eta}) = \lambda\vec{\eta}, \quad P_1(\vec{\xi} + \vec{\eta}) \in \mathcal{D}(B_1), \tag{5.26}$$

where $P_1 : \vec{M}_0(\Omega) \to \vec{H}_1$ is the orthogonal projector on $\vec{H}_1$, while $B_1 := B\big|_{\vec{H}_1} = P_1 B = BP_1 = P_1 B P_1$ is the restriction of the operator B to the subspace $\vec{H}_1$. Obviously, Ker $B_1 = \{0\}$ in $\vec{H}_1$ and its eigenvalues $\lambda_k(B_1)$ are such that

$$\lambda_1(B_1) \leq \ldots \leq \lambda_\kappa(B_1) < 0 < \lambda_{\kappa+1}(B_1) \leq \lambda_{\kappa+j}(B_1) \leq \ldots \tag{5.27}$$

because

$$\lambda_k(B_1) = \lambda_k(B), \quad k = 1, \ldots, \kappa, \quad \lambda_k(B_1) = \lambda_{k+q}(B), \quad k \geq \kappa + 1. \tag{5.28}$$

Lemma 5.2. *Eigenvalues of problem* (5.25)–(5.26) *can transit from the right-hand half-plane to the left-hand one only through the origin.*

Proof. Assume that $\lambda \neq 0$ in (5.25)–(5.26) and represent the latter system as follows:

$$A\vec{\xi} = \lambda(\vec{\xi} + \vec{\eta}), \quad \alpha B_1(\vec{\xi} + \vec{\eta}) = \lambda\vec{\eta}, \quad \vec{\xi} \in \mathcal{D}(A), \quad P_1(\vec{\xi} + \vec{\eta}) \in \mathcal{D}(B_1). \tag{5.29}$$

Applying the operator A^{-1} to the first relation and introducing $\vec{\delta} := \vec{\xi} + \vec{\eta}$, we obtain the equation

$$\vec{\delta} = \lambda A^{-1}\vec{\delta} + \alpha\lambda^{-1}B_1\vec{\delta}. \tag{5.30}$$

Assign $\lambda = i\gamma$, $0 \neq \gamma \in \mathbb{R}$, in the latter equation and multiply it (scalarly in $\vec{M}_0(\Omega)$) by $\vec{\delta}$. Then we see that the left-hand side is equal to $\|\vec{\delta}\|^2$, while the right-hand side is purely imaginary. Hence,

$\vec{\delta} = \vec{0}$. Then (5.29) implies that $\vec{\xi} = \vec{0}$, $\vec{\eta} = \vec{0}$, i.e., problem (5.25)–(5.26) has only the trivial solution if $\lambda = i\gamma$, $0 \neq \gamma \in \mathbb{R}$. $\square$

The latter assertion and Lemma 5.1 imply the following theorem.

Theorem 5.2 (stability change principle)**.** *If a change of physical parameters of the hydraulic system causes a stability loss of normal motions of a capillary viscous fluid, then eigenvalues λ can transit from the right-hand half-plane to the left-hand one only through the origin of the complex plane and only under the condition* $\operatorname{Ker} B_\sigma \neq \{0\}$.

We provide one more property of solutions of problem (5.1)–(5.2) and related problems (5.25)–(5.26) and (5.30).

Lemma 5.3. *Let λ be an eigenvalue of problem* (5.30) *and suppose that one of the following conditions holds*:

(a) $\operatorname{Im}\lambda \neq 0$;

(b) $\operatorname{Im}\lambda = 0$ *and there exists an associated element corresponding to λ.*

Then

$$\operatorname{Re}\lambda > \frac{\lambda_{\min}(\mathrm{A})}{2} > 0. \tag{5.31}$$

The proof of this assertion totally coincides with the corresponding proof of the analogous assertion for a problem of the kind (5.30) for a viscous heavy fluid, where B_1 is a compact nonnegative operator (see, e.g., [16, p. 286–288] and [14, p. 79–81]).

Now we pass from spectral problem (5.25)–(5.26) with unbounded operators A and B_1 to the eigenvalue problem for a compact operator. This is possible because A and B_1 have compact inverse operators.

Lemma 5.4. *If $\lambda \neq 0$, then problem* (5.25)–(5.26) *is equivalent to the problem*

$$\alpha A^{-1}\vec{\xi} + \alpha A^{-1}P_1\vec{\eta} = \mu\vec{\xi}, \quad \mu := \alpha\lambda^{-1}, \tag{5.32}$$

$$-\alpha P_1 A^{-1}\vec{\xi} + (B_1^{-1} - \alpha P_1 A^{-1} P_1)\vec{\eta} = \mu\vec{\eta}. \tag{5.33}$$

Proof. 1. From problem (5.25)–(5.26), represented as (5.29), we see that

$$\vec{\eta} = P_1\vec{\eta} \in \vec{H}_1 \tag{5.34}$$

under the assumption that $\lambda \neq 0$.

Hence, applying the operator A^{-1} to the first relation in (5.29), we obtain (5.32). Then, projecting (5.32) to $\vec{H}_1$, we have

$$\alpha P_1 A^{-1}\vec{\xi} + \alpha P_1 A^{-1} P_1\vec{\eta} = \mu P_1\vec{\xi}. \tag{5.35}$$

From the second equation in (5.29), we see that

$$B^{-1}\vec{\eta} = \mu(P_1\vec{\xi} + \vec{\eta}). \tag{5.36}$$

Substituting the expression for $\mu P_1\vec{\xi}$ from (5.35) to the latter relation, we obtain (5.33).

2. Now we prove that solutions of problem (5.32)–(5.33) satisfy problem (5.25)–(5.26) as well. As above, (5.32) implies (5.35). Then (5.33) implies that $B^{-1}\vec{\eta} - \mu P_1\vec{\xi} = \mu\vec{\eta}$. Then, assuming that $\mu = \dfrac{\alpha}{\lambda}$ and taking into account that $B_1(\vec{\xi} + \vec{\eta}) = B_1(P_1\vec{\xi} + \vec{\eta})$, we obtain (5.26).

Further, it follows from (5.32) that $\vec{\xi} \in \mathcal{D}(A)$ and

$$\alpha(\vec{\xi} + P_1\vec{\eta}) = \mu A\vec{\xi} \Longleftrightarrow A\vec{\xi} = \lambda(\vec{\xi} + \vec{\eta}). \tag{5.37}$$

Using (5.26), we obtain (5.25) from (5.37). $\square$

Taking into account the form of problem (5.32)–(5.33) and properties of operators A^{-1} and B_1^{-1}, we can apply the perturbation theory to this problem, assuming that $\alpha = \nu^{-2}$ is sufficiently small, i.e., the viscosity is sufficiently large. Note that properties of all operators included in Eqs. (5.32)–(5.33) do not depend on ν because internal and external (capillary) forces and the shape of the domain Ω occupied by the fluid do not depend on ν.

Lemma 5.5. *If the positive parameter α of problem* (5.32)*–*(5.33) *satisfies the condition*

$$4\alpha < \frac{\lambda_1(A)}{|\lambda_1(B_1)|}, \tag{5.38}$$

then problem (5.32)*–*(5.33) *has at least κ eigenvalues* (*counted with their multiplicities*) *located in the left-hand half-plane.*

Proof. We treat problem (5.32)–(5.33) as a problem obtained by means of a perturbation of the spectral vector-matrix problem

$$\begin{pmatrix} \alpha A^{-1} & 0 \\ 0 & B_1^{-1} \end{pmatrix} \begin{pmatrix} \vec{\xi} \\ \vec{\eta} \end{pmatrix} = \mu \begin{pmatrix} \vec{\xi} \\ \vec{\eta} \end{pmatrix} \tag{5.39}$$

studied in the Hilbert space $\vec{J}_{0,S}(\Omega) \oplus \vec{M}_0(\Omega)$ by an operator matrix of the kind

$$\alpha T := \alpha \begin{pmatrix} 0 & A^{-1}P_1 \\ -P_1 A^{-1} & -P_1 A^{-1} P_1 \end{pmatrix}. \tag{5.40}$$

Taking into account inequalities (5.27), we see that the spectrum of problem (5.39) consists of the eigenvalues $\{\lambda_k^{-1}(B_1)\}_{k=1}^{\kappa}$ located at the negative semiaxis and of the set $\{\lambda_k^{-1}(A)\}_{k=1}^{\infty} \cup \{\lambda_k^{-1}(B_1)\}_{k=\kappa+1}^{\infty}$ of positive eigenvalues with the limit point at the origin. Obviously, the distance between the greatest eigenvalue $\lambda_1^{-1}(B_1)$ located in the left-hand half-plane and the origin is equal to $|\lambda_1^{-1}(B_1)| = 1/|\lambda_1(B_1)|$.

Therefore, if the norm of perturbation matrix (5.40) is less than $r := 1/(2|\lambda_1(B_1)|)$, then the spectrum of the perturbed problem located in the left-hand half-plane is located in the domain

$$\Lambda_r := \{\lambda \in \mathbb{C} : |\lambda - \tau| < r, \quad \tau \in [\lambda_\kappa^{-1}(B_1), \lambda_1^{-1}(B_1)]\} \tag{5.41}$$

and, perhaps, in the left-hand half-disk of radius r centered at the origin. In particular, the number of eigenvalues of problem (5.32)–(5.33) located in Λ_r is equal to κ.

Let us check whether $\|\alpha T\| = \alpha\|T\| < r$ under condition (5.38). We have the following basic estimate:

$$\left\| \begin{pmatrix} 0 & A^{-1}P_1 \\ -P_1 A^{-1} & -P_1 A^{-1} P_1 \end{pmatrix} \begin{pmatrix} \vec{\xi} \\ \vec{\eta} \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} A^{-1}P_1\vec{\eta} \\ -P_1 A^{-1}\vec{\xi} - P_1 A^{-1} P_1 \vec{\eta} \end{pmatrix} \right\|^2$$

$$\leq \|A^{-1}\|^2 \Big\{ \|\vec{\eta}\|^2 + \Big(\|\vec{\xi}\|^2 + \|\vec{\eta}\|^2 + 2\|\vec{\xi}\|\|\vec{\eta}\| \Big) \Big\} < 4\|A^{-1}\|^2 \Big(\|\vec{\xi}\|^2 + \|\vec{\eta}\|^2 \Big).$$

This implies that $\|\alpha T\| \leq 2\alpha\|A^{-1}\| = \dfrac{2\alpha}{\lambda_1(A)}$; then $\|\alpha T\| < r$ by virtue of (5.38). □

Using Lemma 5.5 and the facts provided above, we obtain the following conclusion.

Theorem 5.3. *Suppose that condition* (5.38) *is satisfied. Then the number of eigenvalues of problem* (5.25)*–*(5.26) *located in the left-hand complex half-plane is equal to κ and those eigenvalues are located at the negative semiaxis.*

The proof of this theorem consists of several steps.

As we have noted above (see Lemma 5.2), problem (5.25)–(5.26) is equivalent to Eq. (5.30) and there are no associated eigenelements corresponding to eigenvalues from the left-hand half-plane (see Lemma 5.3).

Lemma 5.6. *Let λ_k be a negative eigenvalue of problem* (5.30) *and $\{\vec{\delta}_{kj}\}_{j=1}^{\alpha_k}$ be the eigenelements corresponding to λ_k. Then elements $\{P_1\vec{\delta}_{kj}\}_{j=1}^{\alpha_k}$ are linearly independent.*

Proof. From (5.30), we have

$$\vec{\delta}_{kj} = \lambda_k A^{-1}\vec{\delta}_{kj} + \alpha\lambda_k^{-1} B_1\vec{\delta}_{kj}, \quad j = 1, \ldots, \alpha_k. \tag{5.42}$$

Let

$$\sum_{j=1}^{\alpha_k} c_j P_1\vec{\delta}_{kj} = \vec{0}. \tag{5.43}$$

Then (5.42) implies the relation

$$\alpha\lambda_k^{-1}\sum_{j=1}^{\alpha_k} c_j B_1\vec{\delta}_{kj} = \lambda_k\left(I - \lambda_k A^{-1}\right)\sum_{j=1}^{\alpha_k} c_j\vec{\delta}_{kj} = \vec{0} \quad (B_1P_1 = B_1). \tag{5.44}$$

Since $\lambda_k < 0$, it follows that $I - \lambda_k A^{-1} \gg 0$. Therefore, the operator $I - \lambda_k A^{-1}$ has a bounded inverse. Then $\sum_{j=1}^{\alpha_k} c_j\vec{\delta}_{kj} = \vec{0}$. Since $\vec{\delta}_{kj}$ are linearly independent, it follows that $c_j = 0$, $j = 1, \ldots, \alpha_k$. This completes the proof of the lemma. □

The second auxiliary assertion refers to a property of eigenelements corresponding to different eigenvalues.

Lemma 5.7. *Elements $\{P_1\vec{\delta}_k\}_{k=1}^{\kappa}$ corresponding to eigenelements $\{\vec{\delta}_k\}_{k=1}^{\kappa}$ of problem* (5.30) *for different negative eigenvalues λ_k are linearly independent.*

Proof. Let $\lambda < 0$ and $\vec{\delta}$ be a solution of problem (5.30) for that λ. We have

$$\vec{\delta} = \lambda A^{-1}\vec{\delta} + \alpha\lambda^{-1}B_1\vec{\delta}. \tag{5.45}$$

Represent $\vec{\delta}$ as follows:

$$\vec{\delta} = \vec{\delta}_0 + \vec{\delta}_1, \quad \vec{\delta}_0 = P_0\vec{\delta} \in \vec{H}_0, \quad \vec{\delta}_1 = P_1\vec{\delta} \in \vec{H}_1. \tag{5.46}$$

Substitute (5.46) in (5.45) and apply the orthogonal projectors P_0 and P_1 to the obtained relation. This yields the following system of equations:

$$\vec{\delta}_0 = \lambda P_0 A^{-1}(P_0\vec{\delta}_0 + P_1\vec{\delta}_1), \quad \vec{\delta}_1 = \lambda P_1 A^{-1}(P_0\vec{\delta}_0 + P_1\vec{\delta}_1) + \alpha\lambda^{-1}B_1\vec{\delta}_1. \tag{5.47}$$

Since $\lambda < 0$, it follows that the operator $I_0 - \lambda P_0 A^{-1} P_0$ is positive definite in $\vec{H}_0$; therefore, from (5.47), we obtain the equation

$$\vec{\delta}_1 = \lambda\left[\lambda P_1 A^{-1} P_0 (I_0 - \lambda P_0 A^{-1} P_0)^{-1} P_0 A^{-1} P_1 + P_1 A^{-1} P_1\right]\vec{\delta}_1 + \alpha\lambda^{-1}B_1\vec{\delta}_1. \tag{5.48}$$

It follows from the reasoning above that problems (5.48) and (5.45) are equivalent to each other and the linear independence of their eigenelements $\{\vec{\delta}_k\}_{k=1}^{\kappa}$ and $\{P_1\vec{\delta}_k\}_{k=1}^{\kappa}$ is equivalent. □

The proof of Theorem 5.3 is based on Lemmas 5.6 and 5.7. It follows from Lemma 5.3 that all eigenvalues of problem (5.25)–(5.26) located in the left-hand half-plane are real. It follows from Eq. (5.30) that

$$\lambda = \frac{\|\vec{\delta}\|^2}{\|A^{-1/2}\vec{\delta}\|^2 + \alpha|\lambda|^{-2}(B_1\vec{\delta}, \vec{\delta})} \tag{5.49}$$

and the quadratic form $(B_1\vec{\delta}, \vec{\delta}) = (B_1P_1\vec{\delta}, P_1\vec{\delta})$ of the operator B_1 is negative definite on the subspace of dimension κ spanned on the initial κ eigenelements of the operator B_1; the latter assertion is valid due to (5.27) implying that the number of negative second-order terms of the specified quadratic form is equal to κ.

If $\lambda < 0$ in (5.49), then

$$(B_1\vec{\delta}, \vec{\delta}) = (J_\kappa |B_1|^{1/2} P_1\vec{\delta}, |B_1|^{1/2} P_1\vec{\delta}\,) > 0, \tag{5.50}$$

where

$$|B_1| = (B_1^{1/2})^{1/2} \gg 0, \quad J_\kappa = B_1|B_1|^{-1}, \quad B_1 = J_\kappa|B_1|^{-1} = |B_1|^{-1/2} J_\kappa |B_1|^{-1/2}. \tag{5.51}$$

Considering the operator J_κ in the orthogonal decomposition of the space $\vec{H}_1 = \vec{M}_0(\Omega) \ominus \vec{H}_0$, corresponding to negative and positive eigenvalues and eigenelements of the operator B_1 (see (5.27)), i.e., in the decomposition

$$\vec{H}_1 = \vec{H}_{1,-} \oplus \vec{H}_{1,+}, \quad \dim \vec{H}_{1,-} = \kappa, \tag{5.52}$$

we see that $J_\kappa = \operatorname{diag}(-I_-; I_+)$ and $J_\kappa = J_\kappa^{-1} = J_\kappa^*$.

It follows from the above properties that an indefinite scalar product in the Pontryagin space Π_κ determined by the operator J_κ corresponds to the quadratic form in (5.50). Property (5.50) implies that eigenelements $\{P_1\vec{\delta}\}$ of problem (5.30) corresponding to its negative eigenvalues are negative in Π_κ. Since they are linearly independent by virtue of Lemmas 5.6 and 5.7 and the dimension of any negative subspace in Π_κ is equal to κ (see, e.g., [7, p. 73]), it follows that the number of negative eigenvalues of problem (5.30) (counted with their multiplicities) does not exceed κ. By virtue of Lemma 5.5, that number is not less than κ. This implies the assertion of Theorem 5.3.

Now we consider the case where α is arbitrary in problem (5.32)–(5.33), removing condition (5.38).

Theorem 5.4. *Let $\alpha = \alpha_0$ be an arbitrary positive number. Then the number of eigenvalues of problem* (5.32)–(5.33) *located in the left-hand half-plane is equal to κ and those eigenvalues are real.*

Proof. For $\alpha < \alpha_* := \lambda_1(A)/4|\lambda_1(B_1)|$, this is proved in Theorem 5.3.

Now we note that the shape of the domain Ω occupied by the fluid and the properties of the operators in problem (5.32)–(5.33) do not depend on the viscosity ν of the fluid; hence, they do not depend on $\alpha = \nu^{-2}$. Let α_0 be an arbitrary positive value. Then eigenvalues $\mu = \mu(\alpha)$ of problem (5.32)–(5.33) are continuous functions of the parameter α on $(\alpha_*, \alpha_0]$. On the other hand, the specified eigenvalues do not tend to infinity for $\alpha \in (\alpha_*, \alpha_0]$ because eigenvalues $\mu_k(\alpha)$ of the compact operator corresponding to problem (5.32)–(5.33) have a limit point $\mu = 0$.

Also, it follows from (5.32)–(5.33) that $\mu = 0$ is an eigenvalue of the problem for no positive value of α. Indeed, the relations

$$\alpha A^{-1}\vec{\xi} + \alpha A^{-1} P_1\vec{\eta} = 0, \quad B_1^{-1}\vec{\eta} - P_1(\alpha A^{-1}\vec{\xi} + \alpha A^{-1} P_1\vec{\eta}) = \vec{0} \tag{5.53}$$

lead to the trivial solution $\vec{\eta} = \vec{0}$, $\vec{\xi} = \vec{0}$ because $\operatorname{Ker} B_1^{-1} = \{0\}$ and $\operatorname{Ker} A^{-1} = \{0\}$. This implies that eigenvalues $\mu = \mu(\alpha)$ are different from zero for $\alpha \in (\alpha_*, \alpha_0]$. Finally, they cannot get from the left-hand half-plane to the right-hand one along the imaginary axis (see Lemma 5.2). Moreover, they all are located on the real axis (in the left-hand half-plane) for any α. This completes the proof of the theorem. □

All considerations above lead to the following assertion.

Theorem 5.5 (the inversion of Lagrange's theorem on the stability)**.** *Let $\omega_0 = 0$, the potential energy of a capillary viscous fluid in an arbitrary fixed vessel have a rough "nonminimum," and the potential energy operator B_σ satisfy conditions* (5.3). *Then the number of negative eigenvalues of problem* (5.1)–(5.2) (*on normal oscillations of the fluid*) *is equal to κ and zero is its eigenvalue of multiplicity q.*

In particular, if $\kappa \geq 1$ and $q \geq 0$, then the studied hydraulic system is dynamically unstable.

Proof. We use Theorem 5.4 and the fact that problems (5.1)–(5.2) and (5.32)–(5.33) are equivalent to each other for $\lambda \neq 0$ (see Lemma 5.4), while $\mu = \alpha\widetilde{\lambda}^{-1}$, $\alpha = \nu^{-2}$, and $\widetilde{\lambda} = \dfrac{\lambda}{\nu}$ (see (5.1)). □

To conclude the section, we note that eigenvalues $\widetilde{\lambda}$ can transit from the right-hand half-plane to the left-hand one only if $\operatorname{Ker} B_\sigma \neq \{0\}$ (see Lemma 5.1).

5.2. On unstable motions for uniformly rotating fluid. Consider the case where a vessel with the fluid uniformly rotates with angular velocity $\omega_0 \neq 0$. Then we have a more general problem (see (4.1)–(4.2)) instead of (5.1)–(5.2):

$$A[\vec{\xi} - 2i\omega_0\nu^{-1}S(\vec{\xi} + \vec{\eta})] - \alpha B(P\vec{\xi} + \vec{\eta}) = \widetilde{\lambda}\vec{\xi}, \tag{5.54}$$

$$\alpha B(P\vec{\xi} + \vec{\eta}) = \widetilde{\lambda}\vec{\eta}, \quad P\vec{\xi} + \vec{\eta} \in \mathcal{D}(B), \tag{5.55}$$

$$\vec{\xi} - 2i\omega_0\nu^{-1}S(\vec{\xi} + \vec{\eta}) \in \mathcal{D}(A), \quad S := A^{-1/2}S_0A^{-1/2}, \quad \widetilde{\lambda} = \frac{\lambda}{\nu}, \quad \alpha = \nu^{-2}. \tag{5.56}$$

First, we note the following fact.

Lemma 5.8. *For any $\vec{\varphi} \in \mathcal{D}(A)$ and $\vec{\psi} \in \mathcal{D}(B)$, there exist elements $\vec{\xi} \in \vec{J}_{0,S}(\Omega)$ and $\vec{\eta} \in \vec{M}_0(\Omega)$ such that the following conditions are satisfied:*

$$\vec{\xi} - 2i\omega_0 S(\vec{\xi} + \vec{\eta}) = \vec{\varphi}, \quad P\vec{\xi} + \vec{\eta} = \vec{\psi} \in \mathcal{D}(B). \tag{5.57}$$

Proof. From the second equation, we have $\vec{\eta} = \vec{\psi} - P\vec{\xi}$. Substituting it in the first one, we obtain the equation

$$\left(I - 2i\omega_0\nu^{-1}SP_0\right)\vec{\xi} = \vec{\varphi} + 2i\omega_0\nu^{-1}S\vec{\psi}, \quad P_0 = I - P. \tag{5.58}$$

The operator S is compact. Therefore, in order to prove that the operator $I - 2i\omega_0\nu^{-1}SP_0$ has a bounded inverse, it suffices to prove that the equation

$$\left(I - 2i\omega_0\nu^{-1}SP_0\right)\vec{\xi} = \vec{0} \tag{5.59}$$

has only the trivial solution. Represent $\vec{\xi}$ as $\vec{\xi} = \vec{\xi}_0 + \vec{\xi}_1$, where $\vec{\xi}_0 = P_0\vec{\xi}$ and $\vec{\xi}_1 = P\vec{\xi}$. Then (5.59) is equivalent to the two equations

$$\vec{\xi}_0 = 2i\omega_0\nu^{-1}P_0SP_0\vec{\xi}_0, \quad \vec{\xi}_1 = 2i\omega_0\nu^{-1}PSP_0\vec{\xi}_0.$$

Since $S = S^*$, it follows that the former equation has only the trivial solution. Then $\vec{\xi}_0 = \vec{0}$, $\vec{\xi}_1 = \vec{0}$, and $\vec{\xi} = 0$.

Hence, there exists a unique element $\vec{\xi}$ satisfying Eq. (5.58). Therefore, there exists a unique $\vec{\eta} = \vec{\psi} - P\vec{\xi}$. □

The scheme of the proof of the inversion of Lagrange's theorem on the stability for problem (5.54)–(5.56) is the same as in Sec. 5.1. The distinctions in the line of reasoning are negligible.

This proof consists of the following steps.

1. Again, assume that conditions (5.3) are satisfied for the potential energy operator B_σ. Then the assertion of Theorem 5.1 is valid for the operator $B = Q^*B_\sigma Q$.

2. Now we have the following assertion instead of Lemma 5.1.

Lemma 5.9. *Suppose that* $\operatorname{Ker} B \neq \{0\}$ *and, therefore, $q > 0$ in* (5.4). *Then problem* (5.54)–(5.56) *has a transitional solution of the kind*

$$\widetilde{\lambda} = \lambda_0 = 0, \quad \vec{\eta} = \vec{\eta}_0 := \left(P\left(I - 2i\omega_0\nu^{-1}S\right)^{-1}P\right)^{-1}\vec{\psi}, \quad \forall\vec{\psi} \in \operatorname{Ker} B, \tag{5.60}$$

$$\vec{\xi} = \vec{\xi}_0 := 2i\omega_0\nu^{-1}S\left(I - 2i\omega_0\nu^{-1}S\right)^{-1}\vec{\eta}_0,$$

where P is the identity operator in $\vec{M}_0(\Omega)$ and $P\vec{J}_{0,S}(\Omega) = \vec{M}_0(\Omega)$.

Proof. Assuming $\widetilde{\lambda} = 0$ in (5.54)–(5.55), we have

$$A\left(\vec{\xi} - 2i\omega_0\nu^{-1}S(\vec{\xi} + \vec{\eta})\right) = \vec{0}, \quad B(P\vec{\xi} + \vec{\eta}) = \vec{0}. \tag{5.61}$$

From the former equation, we get

$$(I - 2i\omega_0\nu^{-1}S)\vec{\xi} = 2i\omega_0\nu^{-1}S\vec{\eta}.$$

Since $S = S^*$, it follows that there exists a bounded inverse $(I - 2i\omega_0\nu^{-1}S)^{-1}$; then we have

$$\vec{\xi} = 2i\omega_0\nu^{-1}S(I - 2i\omega_0\nu^{-1}S)^{-1}\vec{\eta}. \tag{5.62}$$

From the latter equation in (5.61), we have $P\vec{\xi} + \vec{\eta} = \vec{\psi} \in \operatorname{Ker} B$. Substituting (5.62) in that relation, we get the equation

$$F\vec{\eta} := P(I - 2i\omega_0\nu^{-1}S)^{-1}P\vec{\eta} = \vec{\psi}. \tag{5.63}$$

Now we note that the operator F has a bounded inverse in $\vec{M}_0(\Omega)$. Indeed, if $F\vec{\eta} = 0$, then

$$(F\vec{\eta}, \vec{\eta}) = \left((I - 2i\omega_0\nu^{-1}S)^{-1}\vec{\eta}, \vec{\eta}\right) = 0, \quad \vec{\eta} = P\vec{\eta}.$$

Change the variables: $(I - 2i\omega_0\nu^{-1}S)^{-1}\vec{\eta} =: \vec{v}$; this yields

$$\|\vec{v}\|^2 - 2i\omega_0\nu^{-1}(S\vec{v}, \vec{v}) = 0.$$

Since $S = S^*$, we have $\vec{v} = \vec{0}$. Therefore, $\vec{\eta} = \vec{0}$.

Using the invertibility of F, from (5.63), we obtain (5.60) for $\vec{\eta} = \vec{\eta}_0$. Then the other relation from (5.60) for $\vec{\xi} = \vec{\xi}_0$ follows from (5.62). □

3. Using the same line of reasoning as in Sec. 5.1, from (5.54)–(5.56), we have (instead of system (5.25)–(5.26))

$$A[\vec{\xi} - 2i\omega_0\nu^{-1}S(\vec{\xi} + \vec{\eta})] - \alpha B_1(P_1\vec{\xi} + \vec{\eta}) = \lambda\vec{\xi}, \tag{5.64}$$

$$\alpha B_1(P_1\vec{\xi} + \vec{\eta}) = \lambda\vec{\eta}, \tag{5.65}$$

$$\vec{\xi} - 2i\omega_0\nu^{-1}S(\vec{\xi} + \vec{\eta}) \in \mathcal{D}(A), \quad P_1\vec{\xi} + \vec{\eta} \in \mathcal{D}(B_1); \tag{5.66}$$

as above (i.e., before Lemma 5.1), $\widetilde{\lambda}$ is replaced by λ for simplicity.

Lemma 5.2 and Theorem 5.2 (the stability change principle) are valid for problem (5.64)–(5.66) as well. Indeed, arguing as in the proof of Lemma 5.2, we obtain the following equation instead of Eq. (5.30):

$$\vec{\delta} = \lambda A^{-1}\vec{\delta} + \alpha\lambda^{-1}B_1\vec{\delta} + 2i\omega_0 S\vec{\delta}; \tag{5.67}$$

this equation has only trivial solutions for $\lambda = i\gamma$ and $0 \neq \gamma \in \mathbb{R}$.

4. If $\omega_0 \neq 0$, then Lemma 5.4 is generalized as follows.

Lemma 5.10. *Let*

$$R(\omega_0\nu^{-1}) := I + 2i\omega_0\nu^{-1}U(\omega_0\nu^{-1})SP_1 \tag{5.68}$$

and

$$U(\omega_0\nu^{-1}) := (I - 2i\omega_0\nu^{-1}S)^{-1}. \tag{5.69}$$

Then the operator $R(\omega_0\nu^{-1})$ has a bounded inverse, $R^{-1}(\omega_0\nu^{-1})$ has the structure

$$R^{-1}(\omega_0\nu^{-1}) = I + T_1(\omega_0\nu^{-1}), \quad T_1(\omega_0\nu^{-1}) \in \mathfrak{S}_\infty, \quad T_1(0) = 0, \tag{5.70}$$

and problem (5.64)–(5.66) *is equivalent to the problem*

$$\begin{gathered}\alpha R^{-1}(\omega_0\nu^{-1})U(\omega_0\nu^{-1})A^{-1}\vec{\xi} + R^{-1}(\omega_0\nu^{-1})[2i\omega_0\nu^{-1}U(\omega_0\nu^{-1})SB_1^{-1} + \alpha U(\omega_0\nu^{-1})A^{-1}P_1]\vec{\eta} = \mu\vec{\xi},\\ B_1^{-1}\vec{\eta} - \alpha P_1R^{-1}(\omega_0\nu^{-1})U(\omega_0\nu^{-1})A^{-1}\vec{\xi} - \alpha P_1R^{-1}(\omega_0\nu^{-1})A^{-1}P_1\vec{\eta}\\ -2i\omega_0\nu^{-1}P_1R^{-1}(\omega_0\nu^{-1})U(\omega_0\nu^{-1})SB_1^{-1}\vec{\eta} = \mu\vec{\eta}, \quad \mu = \alpha\lambda^{-1}.\end{gathered} \tag{5.71}$$

Proof. Since $S = S^* = A^{-1/2} S_0 A^{-1/2} \in \mathfrak{S}_\infty$, it follows that the operator $U(\omega_0\nu^{-1})$ is equal to the sum of the identity operator and a compact operator. Therefore, the operator $R(\omega_0\nu^{-1})$ has the same structure. Hence, to prove (5.70), it suffices to check whether the operator $R(\omega_0\nu^{-1})$ is invertible.

Consider the equation $R(\omega_0\nu^{-1})\vec{\xi} = \vec{0}$ and represent ξ as $\vec{\xi} = \vec{\xi_0} + \vec{\xi_1}$, where $\vec{\xi_0} = P_0\vec{\xi} = (I - P_1)\vec{\xi}$ and $\vec{\xi_1} = P_1\vec{\xi}$. Then we have

$$\vec{\xi_0} + \vec{\xi_1} + 2i\omega_0\nu^{-1}(I - 2i\omega_0\nu^{-1}S)^{-1}S\vec{\xi_1} = \vec{0};$$

this yields the equation

$$\vec{\xi_0} + U(\omega_0\nu^{-1})P_1\vec{\xi_1} = \vec{\xi_0} + P_0U(\omega_0\nu^{-1})P_1\vec{\xi_1} + P_1U(\omega_0\nu^{-1})P_1\vec{\xi_1} = \vec{0}.$$

Then we have

$$\vec{\xi_0} + P_0U(\omega_0\nu^{-1})P_1\vec{\xi_1} = \vec{0}, \quad P_1U(\omega_0\nu^{-1})P_1\vec{\xi_1} = \vec{0}. \tag{5.72}$$

However, the operator $P_1U(\omega_0\nu^{-1})P_1$ is invertible in $\vec{H}_1$. This is proved in the same way as for problem (5.63). Therefore, the latter relation in (5.72) implies that $\vec{\xi_1} = \vec{0}$; hence, the former relation in (5.72) implies that $\vec{\xi_0} = \vec{0}$.

Thus, the operator $R(\omega_0\nu^{-1})$ is invertible. Now we prove that problems (5.64)–(5.66) and (5.71) are equivalent to each other.

From (5.64)–(5.65), we have

$$\vec{\xi} = \lambda U(\omega_0\nu^{-1})A^{-1}(\vec{\xi} + \vec{\eta}) + 2i\omega_0\nu^{-1}U(\omega_0\nu^{-1})S\vec{\eta}, \qquad \vec{\eta} = -P\vec{\xi} + \frac{\lambda}{\alpha}B_1^{-1}\vec{\eta}. \tag{5.73}$$

Substituting the expression of $\vec{\eta}$ in the latter term of the former equation and taking into account that $P_1\vec{\eta} = \vec{\eta}$, we have

$$R(\omega_0\nu^{-1})\vec{\xi} = \lambda U(\omega_0\nu^{-1})A^{-1}(\vec{\xi} + P_1\vec{\eta}) + 2i\omega_0\nu^{-1}\lambda\alpha^{-1}U(\omega_0\nu^{-1})SB_1^{-1}\vec{\eta}. \tag{5.74}$$

This implies the former equation in (5.71) (take into account that $\mu = \alpha\lambda^{-1}$). The latter equation in (5.71) goes from the latter equation of (5.73) if we express $\vec{\xi}$ from the former equation of (5.71).

It is easy to see that all transformations above are invertible. This implies the second assertion of the lemma. □

5. Using Lemma 5.10 and facts proved in Sec. 5.1, we establish the main result on the spectrum structure in the left-hand half-plane (for $\omega_0 \neq 0$) for the studied problem.

Theorem 5.6. *Suppose that, for $\omega_0 \neq 0$, the potential energy of a capillary fluid in a vessel has a rough "nonminimum" and conditions* (5.3) *are satisfied. Then the assertions of Theorem* 5.5 *hold and the hydraulic system is dynamically unstable.*

Proof. First, we note that system (5.71) becomes (5.32)–(5.33) for $\omega_0 = 0$ because $R(0) = I$ and $U(0) = I$ (see (5.68)–(5.70)).

Taking this into account, we change ω_0 by $\varepsilon\omega_0$ in (5.71) and assume (formally) that $0 \leq \varepsilon \leq 1$. Then assertions of Theorem 5.5 are valid for $\varepsilon = 0$ (it is assumed that the operators depend on ω_0 as on a parameter). In particular, if $\varepsilon = 0$, then the number of eigenvalues of problem (5.71) located in the left-hand half-plane $\operatorname{Re}\mu < 0$ is equal to κ and all those eigenvalues are located on the real axis.

Eigenvalues located in the left-hand half-plane are continuous functions of $\varepsilon \in (0, 1)$ and the following facts take place (cf. the proof of Theorem 5.4). The specified eigenvalues cannot go to infinity because (5.71) is an eigenvalue problem for a compact operator. Further, it is easy to check whether problem (5.71) has (for $\mu = 0$) the trivial solution for any ε such that $0 \leq \varepsilon \leq 1$, i.e., eigenvalues cannot be equal to zero. Finally, they cannot transit from the left-hand half-plane to the right-hand one by virtue of property 3.

This completes the proof. □

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