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OSCILLATIONS OF STRATIFIED FLUIDS

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Abstract. We study the problem on small motions and normal oscillations of a system of two heavy immiscible stratified fluids partially filling a fixed vessel. The lower fluid is assumed to be viscous, while the upper one is assumed to be ideal. We find sufficient existence conditions for a strong (with respect to the time variable) solution of the initial-boundary value problem describing the evolution of the specified hydraulic system. For the corresponding spectral system, we obtain results about the localization of the spectrum, asymptotic behavior of branches of eigenvalues, and existence of the substantial spectrum of the problem.

CONTENTS

Introduction

Oscillation problems for a stratified fluid occupying a bounded spatial domain are applied in the seiche theory, oil oscillation theory for tankers, and investigations of cryogenic fluids oscillations in closed containers. Instead of any detailed bibliography, we just mention monographs [3–6, 12, 15, 20] and works [9–11, 17–19] treating various aspects of the oscillation theory for such systems. Particularly, in [17], the oscillation problem for immiscible viscous stratified fluid and ideal homogeneous fluids is studied provided that there exists a free surface.

It is known that if a fluid is vertically stratified with respect to its density, then interesting physical phenomena related to floatability forces arise in the above hydraulic systems. In oceans, those forces generate inner inertial waves of large amplitudes, able to lead to a catastrophe. Oscillations leading to the instability of a ship might arise in an oil tanker. In this work, we study the initial-boundary and the spectral problem on small motions of a hydraulic system consisting of two layer of fluids: the lower fluid is viscous, while the upper one is ideal.

1. Small Motions of Partially Dissipative Hydraulic Systems Consisting of Two Immiscible Stratified Fluids

1.1. Mathematical posing of the problem. Consider a fixed vessel partially occupied by a system of two immiscible fluids. Since the fluids are assumed to be heavy, we do not take into account capillary forces. Suppose that the lower (with respect to the gravitational force) domain Ω_1 is occupied by a viscous stratified incompressible fluid with dynamical viscosity coefficient $\mu = \text{const} > 0$, while the upper domain Ω_2 is occupied by an ideal stratified incompressible fluid. We assume that the densities of the viscous and ideal fluids (ρ_1 and ρ_2 respectively) vary along the vertical axis in the quiescence.

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Let $\vec{n}_i$ $(i=1,2)$ be the unit vector normal to $\partial\Omega_i$ $(i=1,2)$ and directed outside Ω_i $(i=1,2)$. Let S_i denote the part of the vessel wall corresponding to the domain Ω_i $(i=1,2)$. Represent $\Gamma=\partial\Omega_2\setminus\overline{S_2}$ as $\Gamma=\Gamma_1\cup\Gamma_2$, where Γ_1 and Γ_2 are the lower and the upper boundaries of the domain Ω_2 respectively. For Ω_1, we have $\Gamma_1=\partial\Omega_1\setminus\overline{S_1}$. Introduce the coordinate system $Ox_1x_2x_3$ such that the direction of Ox_3 is opposite to the direction of the gravitational force and the origin is located at the surface Γ_1. The thickness of the layer of the ideal fluid is denoted by b.

Further, we assume that the boundaries $\partial\Omega_i$ $(i=1,2)$ of the domains Ω_i are Lipschitz.

Consider the main case of a stable stratification of fluids with respect to the densities $\rho_i=\rho_i(x_3)$ $(i=1,2)$:

$$\begin{gathered}0<N^2_{i,\min}\le N_i^2(x_3)\le N^2_{i,\max}=:N^2_{0,i}<\infty,\\ N_i^2(x_3):=-\frac{g\rho_i'(x_3)}{\rho_i(x_3)},\quad \rho_1(0)>0,\quad \rho_2(b)>0\quad(i=1,2).\end{gathered}\tag{1.1}$$

The functions $N_i(x_3)$ $(i=1,2)$ are called the Brunt–Väisälä frequencies or the floatability frequencies. Their physical interpretation is as follows. If a fluid particle located at a level $x_3=\text{const}$ leaves that level, then it oscillates in the stratified fluid with the frequency $N_i(x_3)$.

For the investigated hydraulic system, consider small motions close to the quiescence. Let $\vec{u}_i(t,x)$ $(i=1,2)$ be the velocity fields in the fluids, $\zeta_i=\zeta_i(t,\hat{x})$ $(\hat{x}\in\Gamma_i)$ be the vertical deviations of the free motion fluid surfaces $\Gamma_i(t)$ from Γ_i $(i=1,2)$, $p_i=p_i(t,x)$ $(x\in\Omega_i)$ be the deviations of the pressure fields from the equilibrium ones, and $\widetilde{\rho}_i=\widetilde{\rho}_i(t,x)$ $(x\in\Omega_i)$ be the deviations of the density fields from the original ones $\rho_i(x_3)$ $(i=1,2)$.

The linear posing of the oscillation initial-boundary value problem for the considered hydraulic system is as follows (see, e.g., [9–11]):

$$\begin{gathered}\frac{\partial\vec{u}_1}{\partial t}=\rho_1^{-1}(x_3)(-\nabla p_1+\mu\Delta\vec{u}_1-\widetilde{\rho}_1 g\vec{e}_3)+\vec{f},\quad \operatorname{div}\vec{u}_1=0\quad(\text{in }\Omega_1),\\ \frac{\partial\vec{u}_2}{\partial t}=\rho_2^{-1}(x_3)(-\nabla p_2-\widetilde{\rho}_2 g\vec{e}_3)+\vec{f},\quad \operatorname{div}\vec{u}_2=0\quad(\text{in }\Omega_2),\\ \frac{\partial\widetilde{\rho}_1}{\partial t}+\nabla\rho_1\cdot\vec{u}_1=0\quad(\text{in }\Omega_1),\qquad \frac{\partial\widetilde{\rho}_2}{\partial t}+\nabla\rho_2\cdot\vec{u}_2=0\quad(\text{in }\Omega_2),\end{gathered}\tag{1.2}$$

$$\begin{gathered}\vec{u}_1=\vec{0}\quad(\text{on }S_1),\qquad \vec{u}_2\cdot\vec{n}_2=0\quad(\text{on }S_2),\qquad \zeta_1=\zeta_2\quad(\text{on }\Gamma_1),\\ \frac{\partial\zeta_1}{\partial t}=\vec{u}_1\cdot\vec{n}_1=\vec{u}_2\cdot\vec{n}_1\quad(\text{on }\Gamma_1),\qquad \frac{\partial\zeta_2}{\partial t}=\vec{u}_2\cdot\vec{n}_2\quad(\text{on }\Gamma_2),\end{gathered}\tag{1.3}$$

$$\begin{gathered}p_2=g\rho_2\zeta_2\quad(\text{on }\Gamma_2),\qquad \int_{\Gamma_1}\zeta_1\,d\zeta_1=0,\qquad \int_{\Gamma_2}\zeta_2\,d\zeta_2=0,\\ \mu\left(\frac{\partial(u_1)_k}{\partial x_3}+\frac{\partial(u_1)_3}{\partial x_k}\right)=0\quad(k=1,2,\quad\text{on }\Gamma_1),\\ -p_1+2\mu\frac{\partial(u_1)_3}{\partial x_3}=-p_2-g\Delta\rho_1\zeta_1\quad(\text{on }\Gamma_1),\qquad \Delta\rho_1:=\rho_1(0)-\rho_2(0),\end{gathered}\tag{1.4}$$

$$\begin{gathered}\vec{u}_i(0,x)=\vec{u}_i^{\,0}(x),\quad \widetilde{\rho}_i(0,x)=\widetilde{\rho}_i^{\,0}(x)\quad(x\in\Omega_i,\quad i=1,2),\\ \zeta_i(0,\hat{x})=\zeta_i^0(\hat{x})\quad(\hat{x}\in\Gamma_i,\quad i=1,2).\end{gathered}\tag{1.5}$$

1.2. On orthogonal decompositions of Hilbert spaces of vector-functions. Here we introduce Hilbert function spaces needed for the investigation of problem (1.2)–(1.5).

Let Ω be a domain in $\mathbb{R}^3$. Let its boundary $\partial\Omega$ be equal to $\overline{S}\cup\Gamma$, where Γ is a connected set such that $\operatorname{mes}\Gamma>0$. Let $H^1_\Gamma(\Omega,\rho)$ denote the space of functions from $H^1(\Omega,\rho)$ such that their mean values over Γ are equal to zero; introduce the norm in $H^1_\Gamma(\Omega,\rho)$ as follows:

$$\|p\|^2_{H^1_\Gamma(\Omega,\rho)} = \int\limits_\Omega \rho^{-1}|\nabla p|^2 d\Omega < \infty, \qquad \int\limits_\Gamma p\, d\Gamma = 0. \tag{1.6}$$

The following orthogonal decomposition holds for $H^1_\Gamma(\Omega,\rho)$ (cf. [8, p. 45] for the case where $\rho = \text{const}$):

$$H^1_\Gamma(\Omega,\rho) = H^1_{h,S}(\Omega,\rho) \oplus H^1_{0,\Gamma}(\Omega,\rho), \tag{1.7}$$

where

$$H^1_{h,S}(\Omega,\rho) = \Big\{ p \in H^1_\Gamma(\Omega,\rho) \,|\quad \nabla\cdot(\rho^{-1}\nabla p) = 0 \quad (\text{in } \Omega),$$
$$\rho^{-1}\nabla p\cdot\vec{n} = 0 \quad (\text{on } S), \qquad \int\limits_\Gamma p\, d\Gamma = 0 \Big\},$$
$$H^1_{0,\Gamma}(\Omega,\rho) = \{\, p \in H^1_\Gamma(\Omega,\rho) \,|\quad p = 0 \quad (\text{on } \Gamma)\,\};$$

note that the orthogonality in (1.7) is understood in the sense of the scalar product corresponding to norm (1.6).

Assume that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$, where Γ_1 and Γ_2 are connected sets placed horizontally and such that their measures are different from zero.

Introduce the following set:

$$H^1_{\widehat{h,S}}(\Omega,\rho) := \Big\{ p \in H^1_\Gamma(\Omega,\rho) |\ \nabla\cdot(\rho^{-1}\nabla p) = 0 \text{ (in } \Omega), \quad \rho^{-1}\nabla p\cdot\vec{n} = 0 \text{ (on } S),$$
$$\int\limits_{\Gamma_1} \frac{\partial p}{\partial n}\, d\Gamma_1 = 0, \quad \int\limits_{\Gamma_2} \frac{\partial p}{\partial n}\, d\Gamma_2 = 0, \quad \int\limits_\Gamma p\, d\Gamma = 0 \Big\}. \tag{1.8}$$

Lemma 1.1. *The following orthogonal decomposition is valid*:

$$H^1_{h,S}(\Omega,\rho) = H^1_{\widehat{h,S}}(\Omega,\rho) \oplus \{\alpha\,\varphi_0\}, \tag{1.9}$$

where the dimension of the subspace $\{\alpha\,\varphi_0\}$ *is equal to one, while the function* φ_0 *is a solution of the following boundary-value problem*:

$$\begin{aligned} &\nabla\cdot(\rho^{-1}\nabla\varphi_0) = 0 \quad (\textit{in } \Omega), \quad \rho^{-1}\nabla\varphi_0\cdot\vec{n} = 0 \quad (\textit{on } S), \\ &\varphi_0 = \operatorname{mes}\Gamma_1 \quad (\textit{on } \Gamma_2), \quad \varphi_0 = -\operatorname{mes}\Gamma_2 \quad (\textit{on } \Gamma_1). \end{aligned} \tag{1.10}$$

□

1.3. Method of orthogonal projections. In this section, we investigate initial-boundary value problem (1.2)–(1.5), projecting the Euler equation and the Navier–Stokes equation on the orthogonal subspaces of the weighted Hilbert spaces $\vec{L}_2(\Omega_i,\rho_i)$ $(i = 1,2)$:

$$\|\vec{u}_i\|^2_{\vec{L}_2(\Omega_i,\rho_i)} := \int\limits_{\Omega_i} \rho_i(x)|\vec{u}_i(x)|^2 d\Omega_i, \qquad i = 1,2. \tag{1.11}$$

For the domain Ω_1, we decompose the space of vector fields $\vec{L}_2(\Omega_1,\rho_1)$ into an orthogonal sum (see, e.g., [10]):

$$\vec{L}_2(\Omega_1,\rho_1) = \vec{J}_{0,S_1}(\Omega_1,\rho_1) \oplus \vec{G}_{0,\Gamma_1}(\Omega_1,\rho_1), \quad \vec{J}_{0,S_1}(\Omega_1,\rho_1) = \vec{J}_0(\Omega_1,\rho_1) \oplus \vec{G}_{h,S_1}(\Omega_1,\rho_1), \tag{1.12}$$

where

$$\vec{J}_0(\Omega_1,\rho_1) := \{\, \vec{u} \mid \operatorname{div}\vec{u} = 0 \text{ (in } \Omega_1),\ \vec{u}\cdot\vec{n} = 0 \text{ (on } \partial\Omega_1)\,\},$$
$$\vec{G}_{h,S_1}(\Omega_1,\rho_1) := \{\, \vec{v} \mid \vec{v} = \rho_1^{-1}(x_3)\nabla p,\ \vec{v}\cdot\vec{n}_1 = 0 \text{ (on } S_1),$$

$$\nabla \cdot \vec{v} = 0 \text{ (in } \Omega_1), \int\limits_{\Gamma_1} p\, d\Gamma_1 = 0 \},$$

$$\vec{G}_{0,\Gamma_1}(\Omega_1, \rho_1) := \{ \vec{w} \mid \vec{w} = \rho_1^{-1}(x_3)\nabla\varphi, \ \varphi = 0 \text{ (on } \Gamma_1) \}.$$

The operations $\operatorname{div} \vec{u}$ and $(\vec{u} \cdot \vec{n})_{\partial\Omega}$ are treated in the sense of distributions (see [8, p. 101–102]).

Let the vector-function space

$$\vec{J}^1_{0,S_1}(\Omega_1, \rho_1) := \{ \vec{u} \in \vec{H}^1(\Omega_1, \rho_1) \mid \operatorname{div} \vec{u} = 0 \text{ (in } \Omega_1), \ \vec{u} = \vec{0} \text{ (on } S_1) \} \tag{1.13}$$

correspond to the domain Ω_1. The norm in $\vec{J}^1_{0,S_1}(\Omega_1, \rho_1)$ is introduced as follows:

$$\|\vec{u}\|^2_{1,\Omega_1} := \frac{1}{2} \int\limits_{\Omega_1} \left(\sum_{j,k=1}^{3} \left| \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right|^2 \right) d\Omega_1; \tag{1.14}$$

this norm is equivalent to the norm in $\vec{H}^1(\Omega_1, \rho_1)$. One can prove that $\vec{J}^1_{0,S_1}(\Omega_1, \rho_1)$ is densely imbedded in the space $\vec{J}_{0,S_1}(\Omega_1, \rho_1)$ (see [8, Sec. 2.2.6]).

For the domain Ω_2, decompose the vector field space $\vec{L}_2(\Omega_2, \rho_2)$ in the orthogonal sum:

$$\vec{L}_2(\Omega_2, \rho_2) = \vec{J}_0(\Omega_2, \rho_2) \oplus \vec{G}_{h,S_2}(\Omega_2, \rho_2) \oplus \vec{G}_{0,\Gamma}(\Omega_2, \rho_2), \quad \Gamma = \Gamma_1 \cup \Gamma_2. \tag{1.15}$$

Remark 1.1. It is easy to see that there is a one-to-one correspondence between quasipotentials p_2 from the space $H^1_\Gamma(\Omega_2, \rho_2)$ and quasipotential fields from $\vec{G}_{h,S_2}(\Omega_2, \rho_2)$, while orthogonal decomposition (1.9) generates the decomposition

$$\vec{G}_{h,S_2}(\Omega_2, \rho_2) = \vec{G}_{\widehat{h,S_2}}(\Omega_2, \rho_2) \oplus \{\alpha\, \rho_2^{-1} \nabla\varphi_0\}, \tag{1.16}$$

where $\varphi_0(x)$ is a solution of problem (1.10), while $\vec{G}_{\widehat{h,S_2}}(\Omega_2, \rho_2)$ are the fields from $\vec{G}_{h,S_2}(\Omega_2, \rho_2)$ such that conditions of the kind (1.8) are satisfied for their quasipotentials p_2; this means that

$$\int\limits_{\Gamma_1} \frac{\partial p_2}{\partial n}\, d\Gamma_1 = 0, \quad \int\limits_{\Gamma_2} \frac{\partial p_2}{\partial n}\, d\Gamma_2 = 0, \quad \int\limits_{\Gamma} p_2\, d\Gamma = 0.$$

Thus, from (1.15) and (1.16), we obtain the orthogonal decomposition

$$\vec{L}_2(\Omega_2, \rho_2) = \vec{J}_0(\Omega_2, \rho_2) \oplus \vec{G}_{\widehat{h,S_2}}(\Omega_2, \rho_2) \oplus \{\alpha\, \rho_2^{-1} \nabla\varphi_0\} \oplus \vec{G}_{0,\Gamma}(\Omega_2, \rho_2). \tag{1.17}$$

Let us apply the orthogonal projecting method to problem (1.2)–(1.5), using (1.12) and (1.17).

Assume that the solutions of problem (1.2)–(1.5) and the given functions are smooth functions of the variable t valued in the Hilbert spaces $\vec{L}_2(\Omega_1, \rho_1)$ (in the domain Ω_1) and $\vec{L}_2(\Omega_2, \rho_2)$ (in the domain Ω_2) respectively. Taking this into account, in the sequel, we replace $\partial/\partial t$ by d/dt.

Introduce the orthogonal projectors P_{0,S_1} and P_{0,Γ_1} on the subspaces $\vec{J}_{0,S_1}(\Omega_1, \rho_1)$ and $\vec{G}_{0,\Gamma_1}(\Omega_1, \rho_1)$ respectively. Since the field $\vec{u}_1$ is solenoidal and satisfies the adhesion condition on S_1, we assume that it belongs to the space $\vec{J}_{0,S_1}(\Omega_1, \rho_1)$. More exactly, it belongs to the space $\vec{J}^1_{0,S_1}(\Omega_1, \rho_1)$ (see (1.13)), which is densely imbedded in $\vec{J}_{0,S_1}(\Omega_1, \rho_1)$. Apply the introduced orthogonal projectors P_{0,S_1} and P_{0,Γ_1} to both sides of the equation for the viscous fluid from (1.2). This yields

$$\frac{d\vec{u}_1}{dt} = -\, \rho_1^{-1} \nabla\widetilde{p}_1 + P_{0,S_1}(\rho_1^{-1}\mu\Delta\vec{u}_1) - P_{0,S_1}(\rho_1^{-1} g\widetilde{\rho}_1 \vec{e}_3) + P_{0,S_1}\vec{f}, \tag{1.18}$$

$$\vec{0} = -\, \rho_1^{-1} \nabla\varphi_1 + P_{0,\Gamma_1}(\rho_1^{-1}\mu\Delta\vec{u}_1) - P_{0,\Gamma_1}(\rho_1^{-1} g\widetilde{\rho}_1 \vec{e}_3) + P_{0,\Gamma_1}\vec{f}, \tag{1.19}$$

where $\rho_1^{-1}\nabla\widetilde{p}_1 := P_{0,S_1}(\rho_1^{-1}\nabla p_1)$, $\quad \rho_1^{-1}\nabla\varphi_1 := P_{0,\Gamma_1}(\rho_1^{-1}\nabla p_1)$.

If $\vec{u}_1$ and $\widetilde{\rho}_1$ are known, then the field $\rho_1^{-1}\nabla\varphi_1 \in \vec{G}_{0,\Gamma_1}(\Omega_1,\rho_1)$ is directly found from Eq. (1.19). However, this field is not contained in (1.18). Therefore, in the sequel, we consider the main Eq. (1.18) for the viscous fluid.

Taking into account decomposition (1.17), we introduce the orthogonal projectors P_0, $P_{\widehat{h,S_2}}$, P_φ, and $P_{0,\Gamma}$ on the corresponding subspaces $\vec{J}_0(\Omega_2,\rho_2)$, $\vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2)$, $\{\alpha\,\rho_2^{-1}\nabla\varphi_0\}$, and $\vec{G}_{0,\Gamma}(\Omega_2,\rho_2)$. Since the field is solenoidal and the impermeability condition is satisfied on the solid wall S_2, we assume that $\vec{u}_2 \in \vec{J}_0(\Omega_2,\rho_2) \oplus \vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2) =: \vec{J}_{\widehat{0,S_2}}(\Omega_2,\rho_2)$.

The field $\rho_2^{-1}\nabla p_2$ is quasipotential. Therefore,

$$\rho_2^{-1}\nabla p_2 \in \vec{G}(\Omega_2,\rho_2) := \vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2) \oplus \{\alpha\rho_2^{-1}\,\nabla\varphi_0\} \oplus \vec{G}_{0,\Gamma}(\Omega_2,\rho_2).$$

Represent the fields $\vec{u}_2$ and $\rho_2^{-1}\nabla p_2$ as follows:

$$\vec{u}_2 = \vec{v}_2 + \rho_2^{-1}\nabla\Phi_2, \qquad \rho_2^{-1}\nabla p_2 = \rho_2^{-1}\nabla p_{2,1} + \rho_2^{-1}\nabla p_{2,2} + \alpha(t)\,\rho_2^{-1}\nabla\varphi_0, \tag{1.20}$$

where

$$\begin{gathered}\vec{v}_2 \in \vec{J}_0(\Omega_2,\rho_2), \qquad \rho_2^{-1}\nabla\Phi_2 \in \vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2),\\ \rho_2^{-1}\nabla p_{2,1} \in \vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2), \qquad \rho_2^{-1}\nabla p_{2,2} \in \vec{G}_{0,\Gamma}(\Omega_2,\rho_2).\end{gathered} \tag{1.21}$$

Substitute those representations in the ideal fluid motion equation from (1.2) and apply the orthogonal projectors corresponding to decomposition (1.17) to that motion equation. This yields:

$$\frac{d\vec{v}_2}{dt} = -\,P_0(\rho_2^{-1}g\widetilde{\rho}_2\vec{e}_3) + P_0\vec{f}, \tag{1.22}$$

$$\frac{d}{dt}(\rho_2^{-1}\nabla\Phi_2) = -\,\rho_2^{-1}\nabla p_{2,1} - P_{\widehat{h,S_2}}(\rho_2^{-1}g\widetilde{\rho}_2\vec{e}_3) + P_{\widehat{h,S_2}}\vec{f}, \tag{1.23}$$

$$\alpha(t)\rho_2^{-1}\nabla\varphi_0 = -\,P_\varphi(\rho_2^{-1}g\widetilde{\rho}_2\vec{e}_3) + P_\varphi\vec{f}, \tag{1.24}$$

$$\rho_2^{-1}\nabla p_{2,2} = -\,P_{0,\Gamma}(\rho_2^{-1}g\widetilde{\rho}_2\vec{e}_3) + P_{0,\Gamma}\vec{f}. \tag{1.25}$$

Relations (1.24) and (1.25) show that $\alpha\,\rho_2^{-1}\nabla\varphi_0$ and $\rho_2^{-1}\nabla p_{2,2}$ can be found if the solution $\widetilde{\rho}_2$ is known. On the other hand, those fields are not included in (1.22) and (1.23). Also, note that elements of the subspace $\{\alpha\,\rho_2^{-1}\nabla\varphi_0\}$ satisfy the following conditions:

$$\begin{gathered}\nabla\cdot(\rho_2^{-1}\nabla\varphi_0) = 0 \;\;(\text{in } \Omega_2), \quad \rho_2^{-1}\nabla\varphi_0\cdot\vec{n} = 0 \;\;(\text{on } S_2),\\ \varphi_0 = \alpha\,\mathrm{mes}\,\Gamma_1 \;\;(\text{on } \Gamma_2), \quad \varphi_0 = -\alpha\,\mathrm{mes}\,\Gamma_2 \;\;(\text{on } \Gamma_1).\end{gathered}$$

Therefore, from (1.24), one can find all coefficients α. Then the component $\{\alpha\rho_2^{-1}\nabla\varphi_0\}$ is found from (1.24).

In the sequel, we consider Eqs. (1.22) and (1.23) for the ideal fluid: we take into account trivial relations (1.24) and (1.25).

1.4. Problem posing after separating trivial relations. Finally, taking into account the reductions above, we pose the problem. Let γ_i denote the operator of normal trace on the boundary Γ_i for fields defined in Ω_i, $i=1,2$: $\gamma_i\vec{u}_i := (\vec{u}_i\cdot\vec{n}_i)_{\Gamma_i}$. Recall that the unit vectors $\vec{n}_i$ are orthogonal to $\partial\Omega_i$ and directed outside Ω_i $i=1,2$. Consider kinematic relations from (1.3). By virtue of representations (1.20) and (1.21), we have:

$$\frac{d\zeta_1}{dt} = \gamma_1\vec{u}_1 = \rho_2^{-1}\nabla\Phi_2\cdot\vec{n}_1 \;\;(\text{on } \Gamma_1), \quad \frac{d\zeta_2}{dt} = \rho_2^{-1}\nabla\Phi_2\cdot\vec{n}_2 \;\;(\text{on } \Gamma_2).$$

Now, separating trivial relations (1.19), (1.24), and (1.25), we pose initial-boundary value problem (1.2)–(1.5) as follows:

$$
\begin{aligned}
&\frac{d\vec{u}_1}{dt} = -\rho_1^{-1}\nabla\widetilde{p}_1 + P_{0,S_1}(\rho_1^{-1}\mu\Delta\vec{u}_1) - P_{0,S_1}(\rho_1^{-1}g\widetilde{\rho}_1\vec{e}_3) + P_{0,S_1}\vec{f} \quad (\text{in } \Omega_1),\\
&\frac{d\widetilde{\rho}_1}{dt} + \nabla\rho_1\cdot\vec{u}_1 = 0 \ \ (\text{in } \Omega_1),\\
&\operatorname{div}\vec{u}_1 = 0 \ \ (\text{in } \Omega_1), \quad \vec{u}_1 = \vec{0} \ \ (\text{on } S_1),
\end{aligned}
\tag{1.26}
$$

$$
\begin{aligned}
&\frac{d}{dt}(\rho_2^{-1}\nabla\Phi_2) = -\rho_2^{-1}\nabla p_{2,1} - P_{\widehat{h,S_2}}(\rho_2^{-1}g\widetilde{\rho}_2\vec{e}_3) + P_{\widehat{h,S_2}}\vec{f} \quad (\text{in } \Omega_2),\\
&\frac{d\vec{v}_2}{dt} = -P_0(\rho_2^{-1}g\widetilde{\rho}_2\vec{e}_3) + P_0\vec{f} \quad (\text{in } \Omega_2),\\
&\nabla\cdot(\rho_2^{-1}\nabla\Phi_2) = 0 \quad (\text{in } \Omega_2), \quad \rho_2^{-1}\nabla\Phi_2\cdot\vec{n}_2 = 0 \quad (\text{on } S_2),\\
&\nabla\cdot\vec{v}_2 = \operatorname{div}\vec{v}_2 = 0 \quad (\text{in } \Omega_2), \quad \vec{v}_2\cdot\vec{n}_2 = 0 \quad (\text{on } \partial\Omega_2),\\
&\frac{d\widetilde{\rho}_2}{dt} + \nabla\rho_2\cdot(\rho_2^{-1}\nabla\Phi_2) + \nabla\rho_2\cdot\vec{v}_2 = 0 \quad (\text{in } \Omega_2),
\end{aligned}
\tag{1.27}
$$

$$
\begin{aligned}
&\rho_2^{-1}\nabla\Phi_2\cdot\vec{n}_2 = -\gamma_1\vec{u}_1 = -\frac{d\zeta_1}{dt} \quad (\text{on } \Gamma_1),\\
&\rho_2^{-1}\nabla\Phi_2\cdot\vec{n}_2 = \gamma_2(\rho_2^{-1}\nabla\Phi_2) = \frac{d\zeta_2}{dt} \quad (\text{on } \Gamma_2),
\end{aligned}
\tag{1.28}
$$

$$
\begin{aligned}
&P_{\Gamma_2}p_{2,1} = g\rho_2\zeta_2 \quad (\text{on } \Gamma_2),\\
&\mu\left(\frac{\partial(u_1)_k}{\partial x_3} + \frac{\partial(u_1)_3}{\partial x_k}\right) = 0 \quad (k = 1,2, \quad \text{on } \Gamma_1),\\
&-P_{\Gamma_1}\widetilde{p}_1 + 2\mu\frac{\partial(u_1)_3}{\partial x_3} = -g\Delta\rho_1\zeta_1 - P_{\Gamma_1}p_{2,1} \quad (\text{on } \Gamma_1),\\
&\Delta\rho_1 := \rho_1(0) - \rho_2(0) > 0,
\end{aligned}
\tag{1.29}
$$

$$
\begin{aligned}
&\int_{\Gamma_2} p_{2,1}\,d\Gamma_2 = -\alpha(t)\operatorname{mes}\Gamma_1\operatorname{mes}\Gamma_2,\\
&\int_{\Gamma_1} \widetilde{p}_1\,d\Gamma_1 = \int_{\Gamma_1} p_{2,1}\,d\Gamma_1 - \alpha(t)\operatorname{mes}\Gamma_1\operatorname{mes}\Gamma_2,
\end{aligned}
\tag{1.30}
$$

$$
\begin{aligned}
&\vec{u}_1(0,x) = P_{0,S_1}\vec{u}_1^0(x), \quad \vec{w}_2(0,x) = P_{\widehat{h,S_2}}\vec{u}_2^0(x), \quad \vec{v}_2(0,x) = P_0\vec{u}_2^0(x),\\
&\widetilde{\rho}_i(0,x) = \widetilde{\rho}_i^0(x), \quad \zeta_i(0,\hat{x}) = \zeta_i^0(\hat{x}) \quad (i = 1,2),
\end{aligned}
\tag{1.31}
$$

where P_{Γ_i} denote the orthogonal projectors on $L_{2,\Gamma_i} := L_2(\Gamma_i) \ominus \{1_{\Gamma_i}\}$ $(i = 1,2)$.

Theorem 1.1. *Initial-boundary value problem* (1.2)–(1.5) *is equivalent to the collection of trivial relations* (1.19), (1.24), *and* (1.25) *and* (*nontrivial*) *initial-boundary value problem* (1.26)–(1.31).

1.5. Auxiliary boundary-value problems and their operators. To consider an operator for the investigated problem, we consider certain auxiliary boundary-value problems. They are known problems of mathematical physics so we omit definitions of their solutions and proofs of their properties; we just briefly formulate properties of the operators of those problems.

Auxiliary problem I.

$$
\rho_1^{-1}(x_3)\nabla p - P_{0,S_1}(\rho_1^{-1}\mu\Delta\vec{u}_1) = \vec{f}, \quad \operatorname{div}\vec{u}_1 = 0 \quad (\text{in } \Omega_1), \quad \vec{u}_1 = \vec{0} \quad (\text{on } S_1),
$$

$$
\mu\left(\frac{\partial(u_1)_k}{\partial x_3} + \frac{\partial(u_1)_3}{\partial x_k}\right) = 0 \quad (k = 1,2, \quad \text{on } \Gamma_1), \quad -p + 2\mu\frac{\partial(u_1)_3}{\partial x_3} = 0 \quad (\text{on } \Gamma_1).
$$

This is an analog of the first Kreĭn auxiliary problem (see [8, p. 116]). For any vector $\vec{f}$ from $\vec{J}_{0,S_1}(\Omega_1,\rho_1)$, it has a unique generalized solution $\vec{u}_1=\mu^{-1}A^{-1}\vec{f}$, where A is the operator of problem I. The operator A is unbounded, self-adjoint, and positive definite, $\overline{\mathcal{D}(A)}=\vec{J}_{0,S_1}(\Omega_1,\rho_1)$, and the operator A^{-1} is positive and compact in $\vec{J}_{0,S_1}(\Omega_1,\rho_1)$.

Auxiliary problem II.

$$\nabla\cdot(\rho_1^{-1}\nabla p)=0\quad(\text{in }\ \Omega_1),\quad \rho_1^{-1}\nabla p\cdot\vec{n}_1=0\quad(\text{on }\ S_1),$$
$$\rho_1^{-1}(0)p=\tau_1\quad(\text{on }\ \Gamma_1),\quad \int\limits_{\Gamma_1}\tau_1\,d\Gamma_1=0.$$

This is an analog of the known Zaremba problem for $\rho_1=\text{const}$ (see [8, p. 45]). It has a unique solution $p\in H^1_{\Gamma_1}(\Omega_1,\rho_1)$ if $\tau_1\in H^{\frac{1}{2}}_{\Gamma_1}$.

Auxiliary problem III.

$$\nabla\cdot(\rho_2^{-1}\nabla\Psi_1)=0\quad(\text{ in }\ \Omega_2\,),\quad \rho_2^{-1}\nabla\Psi_1\cdot\vec{n}_2=0\quad(\text{on }\ S_2),$$
$$\rho_2^{-1}(b)\nabla\Psi_1\cdot\vec{n}_2=0\ (\text{on }\ \Gamma_2),\ \ \rho_2^{-1}(0)\nabla\Psi_1\cdot\vec{n}_2=\eta_1\ (\text{on }\ \Gamma_1),\ \ \int\limits_{\Gamma}\Psi_1\,d\Gamma=0.$$

Auxiliary problem IV.

$$\nabla\cdot(\rho_2^{-1}\nabla\Psi_2)=0\quad(\text{in }\ \Omega_2),\quad \rho_2^{-1}\nabla\Psi_2\cdot\vec{n}_2=0\quad(\text{on }\ S_2),$$
$$\rho_2^{-1}(0)\nabla\Psi_2\cdot\vec{n}_2=0\ (\text{on }\ \Gamma_1),\ \ \rho_2^{-1}(b)\nabla\Psi_2\cdot\vec{n}_2=\eta_2\ (\text{on }\ \Gamma_2),\ \ \int\limits_{\Gamma}\Psi_2\,d\Gamma=0.$$

Problems III and IV are Neumann problems. If $\eta_1\in H^{-\frac{1}{2}}_{\Gamma_1}$, then problem III has a unique solution $\Psi_1\in H^1_{\Gamma}(\Omega_2,\rho_2)$. In the same way, if $\eta_2\in H^{-\frac{1}{2}}_{\Gamma_2}$, then problem IV has a unique solution $\Psi_2\in H^1_{\Gamma}(\Omega_2,\rho_2)$ (see [10, p. 737] and [8, p. 45]).

For the solutions of problems III and IV, introduce the following operators:

$$\rho_2^{-1}(0)P_{\Gamma_1}\Psi_1|_{\Gamma_1}=:C_1\eta_1,\quad \rho_2^{-1}(b)P_{\Gamma_2}\Psi_1|_{\Gamma_2}=:C_2\eta_1,$$
$$\rho_2^{-1}(0)P_{\Gamma_1}\Psi_2|_{\Gamma_1}=:C_3\eta_2,\quad \rho_2^{-1}(b)P_{\Gamma_2}\Psi_2|_{\Gamma_2}=:C_4\eta_2.$$

Note that the operator C_1 is self-adjoint, positive, and compact in L_{2,Γ_1}, while the operator C_4 is self-adjoint, positive, and compact in L_{2,Γ_2}.

Auxiliary problem V.

$$\nabla\cdot(\rho_1^{-1}\nabla w_{1,2})=0\ \ (\text{in }\ \Omega_1),\quad \rho_1^{-1}\nabla w_{1,2}\cdot\vec{n}_1=0\quad(\text{on }\ S_1),$$
$$\rho_1^{-1}(0)\nabla w_{1,2}\cdot\vec{n}_1=\eta_0\quad(\text{on }\ \Gamma_1),\quad \int\limits_{\Gamma_1}w_{1,2}\,d\Gamma_1=0.$$

Problem V is a Neumann problem. If $\eta_0\in H^{-\frac{1}{2}}_{\Gamma_1}$, then it has a unique solution $w_{1,2}\in H^1_{\Gamma_1}(\Omega_1,\rho_1)$. For the solution of problem V, introduce the operator

$$\rho_1^{-1}(0)P_{\Gamma_1}w_{1,2}|_{\Gamma_1}=:C_{0,4}\eta_0.$$

The operator $C_{0,4}$ is self-adjoint, positive, and compact in L_{2,Γ_1}.

Introduce the norms in the spaces $H^{\frac{1}{2}}_{\Gamma_1}$ and $H^{\frac{1}{2}}_{\Gamma_2}$ as follows (see, e.g., [7]):

$$\|\tau_1\|_{H^{\frac{1}{2}}_{\Gamma_1}}=\|C_{0,4}^{-\frac{1}{2}}\tau_1\|_{L_{2,\Gamma_1}},\qquad \|\tau_2\|_{H^{\frac{1}{2}}_{\Gamma_2}}=\|C_4^{-\frac{1}{2}}\tau_2\|_{L_{2,\Gamma_2}},\quad \tau_i\in H^{\frac{1}{2}}_{\Gamma_i}.\tag{1.32}$$

The norm in the space $H_{\Gamma_1}^{-\frac{1}{2}}$ can be defined as one of the following equivalent norms:

$$\|\eta_0\|_{H_{\Gamma_1}^{-\frac{1}{2}},I} = \|C_{0,4}^{\frac{1}{2}}\eta_0\|_{L_{2,\Gamma_1}}, \qquad \|\eta_0\|_{H_{\Gamma_1}^{-\frac{1}{2}},II} = \|C_1^{\frac{1}{2}}\eta_0\|_{L_{2,\Gamma_1}}, \quad \eta_0 \in H_{\Gamma_1}^{-\frac{1}{2}}. \tag{1.33}$$

In the sequel, we treat the norm $\|\cdot\|_{H_{\Gamma_1}^{-\frac{1}{2}},I}$ as the main one. Then the norm $\|\cdot\|_{H_{\Gamma_1}^{-\frac{1}{2}},II}$ is its equivalent. Hence, there exists a positive constant d such that

$$\|\cdot\|_{H_{\Gamma_1}^{-\frac{1}{2}},II} \leq d\|\cdot\|_{H_{\Gamma_1}^{-\frac{1}{2}},I}.$$

Auxiliary problem VI.

$$\nabla\cdot(\rho_2^{-1}\nabla\varkappa) = 0 \quad (\text{in } \Omega_2), \quad \rho_2^{-1}\nabla\varkappa\cdot\vec{n}_2 = 0 \quad (\text{on } S_2),$$
$$\rho_2^{-1}(0)\nabla\varkappa\cdot\vec{n}_2 = 0 \quad (\text{on } \Gamma_1), \quad \rho_2^{-1}(b)\varkappa = \tau_2 \quad (\text{on } \Gamma_2).$$

This is an analog of the Zaremba problem. If $\tau_2 \in H^{\frac{1}{2}}(\Gamma_2)$, then it has a unique solution $\varkappa \in H^1(\Omega_2, \rho_2)$.

1.6. Derivation of the system of operator equations. In problem (1.26), we represent the field $\rho_1^{-1}\nabla\widetilde{p}_1$ as follows: $\rho_1^{-1}\nabla\widetilde{p}_1 = \rho_1^{-1}\nabla\widetilde{p}_{1,1} + \rho_1^{-1}\nabla\widetilde{p}_{1,2}$. Then we select a field $\rho_1^{-1}\nabla\widetilde{p}_{1,1}$ such that the field $\vec{u}_1$ is a solution of the following boundary-value problem:

$$\rho_1^{-1}\nabla\widetilde{p}_{1,1} - P_{0,S_1}(\rho_1^{-1}\mu\Delta\vec{u}_1) = -\rho_1^{-1}\nabla\widetilde{p}_{1,2} - P_{0,S_1}(\rho_1^{-1}g\widetilde{\rho}_1\vec{e}_3) + P_{0,S_1}\vec{f} - \frac{d\vec{u}_1}{dt},$$
$$\operatorname{div}\vec{u}_1 = 0 \quad (\text{in } \Omega_1), \quad \vec{u}_1 = \vec{0} \quad (\text{on } S_1),$$
$$\mu\left(\frac{\partial(u_1)_k}{\partial x_3} + \frac{\partial(u_1)_3}{\partial x_k}\right) = 0 \quad (k = 1, 2, \quad \text{on } \Gamma_1),$$
$$-P_{\Gamma_1}\widetilde{p}_{1,1} + 2\mu\frac{\partial(u_1)_3}{\partial x_3} = 0 \quad (\text{on } \Gamma_1).$$

Using auxiliary problem I, we conclude that the latter boundary-value problem has the unique generalized solution

$$\vec{u}_1 = \mu^{-1}A^{-1}\left(P_{0,S_1}\vec{f} - \frac{d\vec{u}_1}{dt} - \rho_1^{-1}\nabla\widetilde{p}_{1,2} - P_{0,S_1}(\rho_1^{-1}g\widetilde{\rho}_1\vec{e}_3)\right)$$

if the right-hand side belongs to $\vec{J}_{0,S_1}(\Omega_1, \rho_1)$. The properties of the operator A are given in the previous section. This yields:

$$\frac{d\vec{u}_1}{dt} + \mu A\vec{u}_1 + \rho_1^{-1}\nabla\widetilde{p}_{1,2} + P_{0,S_1}(\rho_1^{-1}g\widetilde{\rho}_1\vec{e}_3) = P_{0,S_1}\vec{f} \quad (\text{in } \Omega_1). \tag{1.34}$$

The third condition in (1.29) is split and the following condition for the normal tension on the free boundary of the viscous fluid remains:

$$P_{\Gamma_1}\widetilde{p}_{1,2} = g\Delta\rho_1\zeta_1 + P_{\Gamma_1}p_{2,1} \quad (\text{on } \Gamma_1).$$

Taking into account that $\rho_1^{-1}\nabla\widetilde{p}_{1,2} \in \vec{G}_{h,S_1}(\Omega_1, \rho_1)$, we see that the potential $P_{\Gamma_1}\widetilde{p}_{1,2}$ is a solution of auxiliary problem II for $\tau_1 = g\Delta\rho_1\zeta_1 + P_{\Gamma_1}p_{2,1}$. Therefore,

$$\rho_1^{-1}\nabla\widetilde{p}_{1,2} =: \rho_1^{-1}(0)G_1(g\Delta\rho_1\zeta_1 + P_{\Gamma_1}p_{2,1}), \tag{1.35}$$

where G_1 is a bounded operator from the space $H_{\Gamma_1}^{1/2}$ to the space $\vec{G}_{h,S_1}(\Omega_1, \rho_1)$; this will be proved below.

Introduce the following notation:

$$P_{0,S_1}(\rho_1^{-1}g\widetilde{\rho}_1\vec{e}_3) =: \widetilde{C}_1\widetilde{\rho}_1, \quad -\nabla\rho_1\cdot\vec{u}_1 =: \widetilde{C}_1^*\vec{u}_1. \tag{1.36}$$

Introduce the Hilbert space $\mathfrak{L}_2(\Omega_1)$ of scalar functions with the following scalar product:

$$(\varphi,\psi)_{\mathfrak{L}_2(\Omega_1)} := \int\limits_{\Omega_1} g^2\left[\rho_1(x_3)N_1^2(x_3)\right]^{-1}\varphi(x)\overline{\psi(x)}\,d\Omega_1.$$

Lemma 1.2. *The operators* $\widetilde{C}_1 : \mathfrak{L}_2(\Omega_1)\to \vec{J}_{0,S_1}(\Omega_1,\rho_1)$ *and* $\widetilde{C}_1^* : \vec{J}_{0,S_1}(\Omega_1,\rho_1)\to\mathfrak{L}_2(\Omega_1)$ *defined by relations* (1.36) *are adjoint to each other and*

$$\|\widetilde{C}_1\| = \|\widetilde{C}_1^*\| \le N_{0,1}. \tag{1.37}$$

The proof of this lemma is similar to the proof of the corresponding lemma in [9].

Taking into account (1.35) and (1.36), we represent (1.34) as follows:

$$\frac{d\vec{u}_1}{dt} + \mu A\vec{u}_1 + \rho_1^{-1}(0)g\Delta\rho_1 G_1\zeta_1 + \rho_1^{-1}(0)G_1P_{\Gamma_1}p_{2,1} + \widetilde{C}_1\widetilde{\rho}_1 = P_{0,S_1}\vec{f}. \tag{1.38}$$

Consider ideal fluid equations from (1.26).

Introduce the following notation:

$$\begin{aligned} &P_{\widehat{h,S_2}}(\rho_2^{-1}g\widetilde{\rho}_2\vec{e}_3) =: \widetilde{C}_{2,1}\widetilde{\rho}_2, \quad -\nabla\rho_2\cdot(\rho_2^{-1}\nabla\Phi_2) = \widetilde{C}_{2,1}^*(\rho_2^{-1}\nabla\Phi_2);\\ &P_0(\rho_2^{-1}g\widetilde{\rho}_2\vec{e}_3) =: \widetilde{C}_{2,0}\widetilde{\rho}_2, \quad -\nabla\rho_2\cdot\vec{v}_2 = \widetilde{C}_{2,0}^*\vec{v}_2.\end{aligned} \tag{1.39}$$

Introduce the Hilbert space $\mathfrak{L}_2(\Omega_2)$ of scalar functions with the scalar product

$$(\varphi,\psi)_{\mathfrak{L}_2(\Omega_2)} := \int\limits_{\Omega_2} g^2\left[\rho_2(x_3)N_2^2(x_3)\right]^{-1}\varphi(x)\overline{\psi(x)}\,d\Omega_2.$$

Lemma 1.3. *The operators*

$$\widetilde{C}_{2,1}\colon \mathfrak{L}_2(\Omega_2)\to\vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2) \quad \textit{and} \quad \widetilde{C}_{2,1}^*\colon \vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2)\to\mathfrak{L}_2(\Omega_2)$$

and

$$\widetilde{C}_{2,0}\colon \mathfrak{L}_2(\Omega_2)\to\vec{J}_0(\Omega_2,\rho_2) \quad \textit{and} \quad \widetilde{C}_{2,0}^*\colon \vec{J}_0(\Omega_2,\rho_2)\to\mathfrak{L}_2(\Omega_2)$$

are adjoint to each other and

$$\|\widetilde{C}_{2,1}\| = \|\widetilde{C}_{2,1}^*\| \le N_{0,2}, \quad \|\widetilde{C}_{2,0}\| = \|\widetilde{C}_{2,0}^*\| \le N_{0,2}. \tag{1.40}$$

The proof of this lemma is similar to the proof of Lemma 1.2.

Taking into account the introduced operators, we represent the ideal fluid equations from (1.26) as follows:

$$\begin{aligned} &\frac{d}{dt}(\rho_2^{-1}\nabla\Phi_2) + \rho_2^{-1}\nabla p_{2,1} + \widetilde{C}_2\widetilde{\rho}_2 = P_{\widehat{h,S_2}}\vec{f} \quad (\text{in } \Omega_2),\\ &\frac{d\vec{v}_2}{dt} + \widetilde{C}_{2,0}\widetilde{\rho}_2 = P_0\vec{f} \quad (\text{in } \Omega_2),\\ &\frac{d\widetilde{\rho}_2}{dt} - \widetilde{C}_{2,1}^*(\rho_2^{-1}\nabla\Phi_2) - \widetilde{C}_{2,0}^*\vec{v}_2 = 0 \quad (\text{in } \Omega_2).\end{aligned} \tag{1.41}$$

Remark 1.2. Since $\rho_2^{-1}\nabla\Phi_2\in\vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2)$, taking into account the definition of the space $\vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2)$, one can use solutions of problems III and IV to represent the potential Φ_2 as follows:

$$\Phi_2 = \Psi_1 + \Psi_2, \tag{1.42}$$

where $\eta_1 = -\gamma_1\vec{u}_1$ (on Γ_1), $\eta_2 = \gamma_2(\rho_2^{-1}\nabla\Phi_2)$ (on Γ_2). Then we use a bounded operator to represent the field $\rho_2^{-1}\nabla\Psi_1$ via $\vec{u}_1$. Thus, we decompose the space $\vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2)$ as the following direct sum:

$$\vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2) = \vec{G}_1(\Omega_2,\rho_2)\dotplus\vec{G}_2(\Omega_2,\rho_2), \tag{1.43}$$

where

$$\vec{G}_1(\Omega_2,\rho_2) := \Big\{\rho_2^{-1}\nabla p \,|\, \nabla\cdot(\rho_2^{-1}\nabla p) = 0 \ \text{(in } \Omega_2), \quad \rho_2^{-1}\nabla p\cdot\vec{n}_2 = 0 \ \text{(on } S_2),$$

$$\rho_2^{-1}\nabla p\cdot\vec{n}_2 = 0 \ \text{(on } \Gamma_2), \quad \int\limits_\Gamma p\,d\Gamma = 0\Big\},$$

$$\vec{G}_2(\Omega_2,\rho_2) =: \Big\{\rho_2^{-1}\nabla p \,|\, \nabla\cdot(\rho_2^{-1}\nabla p) = 0 \ \text{(in } \Omega_2), \quad \rho_2^{-1}\nabla p\cdot\vec{n}_2 = 0 \ \text{(on } S_2),$$

$$\rho_2^{-1}\nabla p\cdot\vec{n}_2 = 0 \ \text{(on } \Gamma_1), \quad \int\limits_\Gamma p\,d\Gamma = 0\Big\}.$$

Remark 1.3. Note that the operator equations we are trying to obtain include the normal trace of the field $\rho_2^{-1}\nabla\Phi_2$ on the boundary Γ_2. By virtue of decomposition (1.43), the normal traces of the fields $\rho_2^{-1}\nabla\Phi_2$ and $\rho_2^{-1}\nabla\Psi_2$ coincide with each other on the boundary Γ_2. In the sequel, we use the notation $\vec{w}_2 := \rho_2^{-1}\nabla\Psi_2$. Then the field $\rho_2^{-1}\nabla\Phi_2$ is decomposed as follows:

$$\rho_2^{-1}\nabla\Phi_2 = \rho_2^{-1}\nabla\Psi_1 + \vec{w}_2, \quad \rho_2^{-1}\nabla\Psi_1 \in \vec{G}_1(\Omega_2,\rho_2), \quad \vec{w}_2 \in \vec{G}_2(\Omega_2,\rho_2). \tag{1.44}$$

From the first equation of (1.41), we obtain the Cauchy–Lagrange integral:

$$\frac{d}{dt}\Phi_2 + p_{2,1} + \Psi = F + C(t) \quad \text{(in } \Omega_2), \tag{1.45}$$

where $\widetilde{C}_{2,1}\widetilde{\rho}_2 =: \rho_2^{-1}\nabla\Psi$ and $P_{\widehat{h,S_2}}\vec{f} =: \rho_2^{-1}\nabla F$. Consider that equation on Γ_2 and project it on L_{2,Γ_2}. Then the dynamical condition $P_{\Gamma_2}p_{2,1} = g\rho_2\zeta_2$ (on Γ_2) takes the following form:

$$-P_{\Gamma_2}\frac{d}{dt}\Phi_2 - P_{\Gamma_2}\Psi + P_{\Gamma_2}F = g\rho_2\zeta_2 \quad \text{(on } \Gamma_2). \tag{1.46}$$

Let us represent $P_{\Gamma_2}(\Phi_2|_{\Gamma_2})$ by means of representation (1.42) and the operators C_k:

$$P_{\Gamma_2}(\Phi_2|_{\Gamma_2}) = P_{\Gamma_2}(\Psi_1|_{\Gamma_2}) + P_{\Gamma_2}(\Psi_2|_{\Gamma_2}) = \rho_2(b)(-C_2\gamma_1\vec{u}_1 + C_4\gamma_2\vec{w}_2) \quad \text{(on } \Gamma_2);$$

then (1.46) takes the following form:

$$-\rho_2(b)\frac{d}{dt}(-C_2\gamma_1\vec{u}_1 + C_4\gamma_2\vec{w}_2) - P_{\Gamma_2}\Psi + P_{\Gamma_2}F = g\rho_2(b)\zeta_2. \tag{1.47}$$

Coming back to auxiliary problem VI, we note that if $\tau_2 = \text{const}$, then problem VI has the unique solution $\varkappa = \text{const}$. Therefore, the function

$$\widetilde{\varkappa} := \varkappa - \frac{1}{\operatorname{mes}\Gamma_1 + \operatorname{mes}\Gamma_2}\int\limits_\Gamma \varkappa\,d\Gamma$$

is the unique solution of problem VI for

$$\widetilde{\tau} := \tau - \frac{1}{\operatorname{mes}\Gamma_1 + \operatorname{mes}\Gamma_2}\int\limits_\Gamma \tau\,d\Gamma$$

and it is easy to check that $\widetilde{\varkappa} \in H^1_\Gamma(\Omega_2,\rho_2)$. One can consider that

$$\rho_2^{-1}\nabla\widetilde{\varkappa} = \rho_2^{-1}\nabla\varkappa = G_2\tau_2, \quad \rho_2^{-1}\nabla\widetilde{\varkappa} \in \vec{G}_2(\Omega_2,\rho_2). \tag{1.48}$$

The operator G_2 is an isometric operator from the space $H^{\frac{1}{2}}_{\Gamma_2}$ to the space $\vec{G}_2(\Omega_2,\rho_2)$. This will be proved below.

Apply the operator G_2 from (1.48) to both sides of Eq. (1.47). This yields one of the sought equations such that its terms belong to the space $\vec{G}_2(\Omega_2, \rho_2)$:

$$-\rho_2(b)\frac{d}{dt}(-G_2C_2\gamma_1\vec{u}_1 + G_2C_4\gamma_2\vec{w}_2) - G_2P_{\Gamma_2}\Psi + G_2P_{\Gamma_2}F = g\rho_2(b)G_2\zeta_2. \tag{1.49}$$

Consider the Cauchy–Lagrange integral from (1.45) on Γ_1 and project it on L_{2,Γ_1}:

$$P_{\Gamma_1}p_{2,1} = -P_{\Gamma_1}\frac{d}{dt}\Phi_2 - P_{\Gamma_1}\Psi + P_{\Gamma_1}F \quad (\text{on } \Gamma_1). \tag{1.50}$$

Express $P_{\Gamma_1}(\Phi_2|_{\Gamma_1})$, using representation (1.42) and the operators C_k:

$$P_{\Gamma_1}(\Phi_2|_{\Gamma_1}) = P_{\Gamma_1}(\Psi_1|_{\Gamma_1}) + P_{\Gamma_1}(\Psi_2|_{\Gamma_1}) = \rho_2(0)(-C_1\gamma_1\vec{u}_1 + C_3\gamma_2\vec{w}_2). \tag{1.51}$$

Substitute expressions (1.50) and (1.51) in Eq. (1.38):

$$\begin{aligned}\frac{d\vec{u}_1}{dt} + \mu A\vec{u}_1 + \rho_1^{-1}(0)g\Delta\rho_1G_1\zeta_1 + \widetilde{C}_1\widetilde{\rho}_1 - \rho_1^{-1}(0)\rho_2(0)\frac{d}{dt}(-G_1C_1\gamma_1\vec{u}_1 + G_1C_3\gamma_2\vec{w}_2)&\\ - \rho_1^{-1}(0)G_1P_{\Gamma_1}\Psi + \rho_1^{-1}(0)G_1P_{\Gamma_1}F = P_{0,S_1}\vec{f} \quad (\text{in } \Omega_1).&\end{aligned}$$

This is a sought operator equation as well.

The results of this subsection can be formulated as the following lemma.

Lemma 1.4. *Any classical solution of system* (1.26)–(1.31) *satisfies the following system of differential equations*:

$$\begin{aligned}&\frac{d}{dt}\left(\vec{u}_1 + \rho_1^{-1}(0)\rho_2(0)(G_1C_1\gamma_1\vec{u}_1 - G_1C_3\gamma_2\vec{w}_2)\right) + \mu A\vec{u}_1 + \widetilde{C}_1\widetilde{\rho}_1\\ &\qquad + \rho_1^{-1}(0)\left(g\Delta\rho_1G_1\zeta_1 - G_1P_{\Gamma_1}\Psi + G_1P_{\Gamma_1}F\right) = P_{0,S_1}\vec{f} \quad (\textit{in } \Omega_1),\\ &\frac{d}{dt}\left(-G_2C_2\gamma_1\vec{u}_1 + G_2C_4\gamma_2\vec{w}_2\right) + \rho_2^{-1}(b)G_2P_{\Gamma_2}\Psi\\ &\qquad\qquad + gG_2\zeta_2 = \rho_2^{-1}(b)G_2P_{\Gamma_2}F \quad (\textit{in } \Omega_2),\\ &\frac{d}{dt}\vec{v}_2 + \widetilde{C}_{2,0}\widetilde{\rho}_2 = P_0\vec{f} \quad (\textit{in } \Omega_2),\\ &\rho_1^{-1}(0)g\Delta\rho_1\frac{d}{dt}\zeta_1 - \rho_1^{-1}(0)g\Delta\rho_1\gamma_1\vec{u}_1 = 0 \quad (\textit{on } \Gamma_1),\\ &g\frac{d}{dt}\zeta_2 - g\gamma_2\vec{w}_2 = 0 \quad (\textit{on } \Gamma_2),\\ &\frac{d}{dt}\widetilde{\rho}_1 - \widetilde{C}_1^*\vec{u}_1 = 0 \quad (\textit{in } \Omega_1),\\ &\frac{d}{dt}\widetilde{\rho}_2 - \widetilde{C}_{2,1}^*\left(\rho_2^{-1}\nabla\Psi_1 + \vec{w}_2\right) - \widetilde{C}_{2,0}^*\vec{v}_2 = 0 \quad (\textit{in } \Omega_2),\\ &\vec{u}_1(0,x) = P_{0,S_1}\vec{u}_1^0(x),\ \vec{w}_2(0,x) = \Pi_2P_{\widehat{h,S_2}}\vec{u}_2^0(x),\ \vec{v}_2(0,x) = P_0\vec{u}_2^0(x),\\ &\zeta_i(0,\hat{x}) = \zeta_i^0(\hat{x}), \quad \widetilde{\rho}_i(0,x) = \widetilde{\rho}_i^0(x) \quad (i = 1, 2).\end{aligned} \tag{1.52}$$

By virtue of decomposition (1.42)*, the initial-value data should satisfy the following kinematic condition*:

$$\gamma_1P_{0,S_1}\vec{u}_1^0(x) = -\widetilde{\gamma}_2\Pi_1P_{\widehat{h,S_2}}\vec{u}_2^0(x) \quad (\textit{on } \Gamma_1). \tag{1.53}$$

In this lemma, Π_1 denotes the projector on the subspace $\vec{G}_1(\Omega_2, \rho_2)$, Π_2 denotes the projector on the subspace $\vec{G}_2(\Omega_2, \rho_2)$, and $\widetilde{\gamma}_2$ denotes the operator of normal trace on the boundary Γ_1 for fields defined in the domain Ω_2.

1.7. Reduction of the system to a single operator-differential equation. Properties of operator blocks. It is convenient to consider system (1.52) in the orthogonal sum of Hilbert spaces $\widetilde{H} := \vec{J}_{0,S}(\Omega,\rho) \oplus H \oplus \mathfrak{L}_2(\Omega)$ as follows:

$$\begin{pmatrix} K & 0 & 0 \\ 0 & g\widetilde{B} & 0 \\ 0 & 0 & \widetilde{C} \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \vec{u} \\ \zeta \\ \widetilde{\rho} \end{pmatrix} + \begin{pmatrix} \mu A_1 & gF_1 & E_1 \\ gF_2 & 0 & 0 \\ E_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ \zeta \\ \widetilde{\rho} \end{pmatrix} = \begin{pmatrix} \vec{f}_1 \\ 0 \\ 0 \end{pmatrix}, \tag{1.54}$$

where $\vec{J}_{0,S}(\Omega,\rho) := \vec{J}_{0,S_1}(\Omega_1,\rho_1) \oplus \vec{G}_2(\Omega_2,\rho_2) \oplus \vec{J}_0(\Omega_2,\rho_2)$, $H := H_1 \oplus H_2$, $H_i = L_2(\Gamma_i) \ominus \{1_i\}$ $(i = 1,2)$, $\mathfrak{L}_2(\Omega) := \mathfrak{L}_2(\Omega_1) \oplus \mathfrak{L}_2(\Omega_2)$, and $(\vec{u};\zeta;\widetilde{\rho})^t = (\vec{u}_1, \vec{w}_2, \vec{v}_2; \zeta_1, \zeta_2; \widetilde{\rho}_1, \widetilde{\rho}_2)^t$. Before we describe the structure of operators contained in (1.54), we note that, by virtue of decomposition (1.42), we have

$$\rho_2^{-1}\nabla\Phi_2 = \rho_2^{-1}\nabla\Psi_1 + \rho_2^{-1}\nabla\Psi_2 = \rho_2^{-1}\nabla\Psi_1 + \vec{w}_2,$$
$$\rho_2^{-1}\nabla\Psi_1 \in \vec{G}_1(\Omega_2,\rho_2), \quad \vec{w}_2 \in \vec{G}_2(\Omega_2,\rho_2).$$

Introduce an operator D as follows:

$$D\vec{u}_1 := \rho_2^{-1}\nabla\Psi_1 \tag{1.55}$$

(its boundedness is proved below in Lemma 1.10).

Using the operator D, we can express the field $\rho_2^{-1}\nabla\Psi_1$ via $\vec{u}_1$.

Taking into account that $\widetilde{C}_{2,1}\tilde{\rho}_2 = \rho_2^{-1}\nabla\Psi$, we introduce an operator B as follows: $B\tilde{\rho}_2 := \Psi$.

Thus, we have:

$$K := \begin{pmatrix} K_1 & K_2 & 0 \\ K_3 & K_4 & 0 \\ 0 & 0 & I_0 \end{pmatrix} = \begin{pmatrix} I_1 + \rho_1^{-1}(0)\rho_2(0)G_1C_1\gamma_1 & -\rho_1^{-1}(0)\rho_2(0)G_1C_3\gamma_2 & 0 \\ -G_2C_2\gamma_1 & G_2C_4\gamma_2 & 0 \\ 0 & 0 & I_0 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} \widetilde{C}_1 & -\rho_1^{-1}(0)G_1P_{\Gamma_1}B \\ 0 & \rho_2^{-1}(b)G_2P_{\Gamma_2}B \\ 0 & \widetilde{C}_{2,0} \end{pmatrix}, \quad E_2 = \begin{pmatrix} -\widetilde{C}_1^* & 0 & 0 \\ -\widetilde{C}_{2,1}^*D & -\widetilde{C}_{2,1}^* & -\widetilde{C}_{2,0}^* \end{pmatrix}, \tag{1.56}$$

$$F_1 = \begin{pmatrix} \rho_1^{-1}(0)\Delta\rho_1G_1 & 0 \\ 0 & G_2 \\ 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} -\rho_1^{-1}(0)\Delta\rho_1\gamma_1 & 0 & 0 \\ 0 & -\gamma_2 & 0 \end{pmatrix}.$$

The other blocks have the diagonal form:

$$\widetilde{B} = \operatorname{diag}(\rho_1^{-1}(0)\Delta\rho_1 I_{\Gamma_1}, I_{\Gamma_2}), \quad A_1 = \operatorname{diag}(A,0,0), \quad \widetilde{C} = \operatorname{diag}(I_{\mathfrak{L}_2(\Omega_1)}, I_{\mathfrak{L}_2(\Omega_2)}).$$

The right-hand side of (1.54) has the following form:

$$(\vec{f}_1;0;0)^t = \left(P_{0,S_1}\vec{f} - \rho_1^{-1}(0)G_1P_{\Gamma_1}F, \rho_2^{-1}(b)G_2P_{\Gamma_2}F, P_0\vec{f}; 0; 0\right)^t.$$

Initial-value conditions (1.52) for Eq. (1.54) can be written in a shorter way:

$$(\vec{u}^0;\zeta^0,\widetilde{\rho}^0)^t = (P_{0,S_1}\vec{u}_1^0(x), \Pi_2P_{\widehat{h,S_2}}\vec{u}_2^0(x), P_0\vec{u}_2^0(x); \zeta_1^0(\widehat{x}), \zeta_2^0(\widehat{x}); \widetilde{\rho}_1^0(x), \widetilde{\rho}_2^0(x))^t. \tag{1.57}$$

The main result of the subsection is as follows:

Theorem 1.2. *Any classical solution of problem* (1.26)–(1.31) *satisfies the Cauchy problem* (1.54), (1.57) *in the Hilbert space* $\widetilde{H}$.

Now we investigate properties of operator blocks in problem (1.54).

Lemma 1.5. *The operator* G_2 *is an isometric operator from the space* $H_{\Gamma_2}^{\frac{1}{2}}$ *to the space* $\vec{G}_2(\Omega_2,\rho_2)$. *The operator* G_1 *is an isometric operator from the space* $H_{\Gamma_1}^{\frac{1}{2}}$ *to the space* $\vec{G}_{h,S_1}(\Omega_1,\rho_1)$.

Proof. Let $\rho_2^{-1}\nabla\varkappa$ denote the result of the action of the operator G_2. Then, for any $\tau_2 \in H_{\Gamma_2}^{\frac{1}{2}}$, we have

$$\|G_2\tau_2\|^2_{\vec{G}_2(\Omega_2,\rho_2)} = \int\limits_{\Omega_2} \rho_2^{-1}(x_3)|\nabla\varkappa|^2\, d\Omega_2 = \int\limits_{\Omega_2} \nabla\varkappa \cdot \overline{\rho_2^{-1}(x_3)\nabla\varkappa}\, d\Omega_2$$
$$= -\int\limits_{\Omega_2} \varkappa \cdot \overline{\nabla\cdot(\rho_2^{-1}(x_3)\nabla\varkappa)}\, d\Omega_2 + \int\limits_{\partial\Omega_2} \varkappa \cdot \overline{\rho_2^{-1}(x_3)\nabla\varkappa\cdot\vec{n}_2}\, dS$$
$$= \int\limits_{\Gamma_2} \rho_2(b)\tau_2 \cdot \overline{\rho_2^{-1}(b)\nabla\varkappa\cdot\vec{n}_2}\, d\Gamma_2 = \int\limits_{\Gamma_2} \rho_2(b)\tau_2\overline{C_4^{-1}\tau_2}\, d\Gamma_2 = \|C_4^{-\frac{1}{2}}\tau_2\|^2_{H_2} = \|\tau_2\|^2_{H_{\Gamma_2}^{\frac{1}{2}}}.$$

This implies that G_2 is an isometric operator from the space $H_{\Gamma_2}^{\frac{1}{2}}$ to the space $\vec{G}_2(\Omega_2,\rho_2)$. For the operator G_1, the similar property is proved in the same way. This completes the proof. □

Lemma 1.6. *For $\gamma_1\colon \vec{J}_{0,S_1}(\Omega_1,\rho_1) \to H_{\Gamma_1}^{-\frac{1}{2}}$ and $G_1\colon H_{\Gamma_1}^{\frac{1}{2}} \to \vec{G}_{h,S_1}(\Omega_1,\rho_1)$, the operators γ_1 and G_1 are adjoint to each other.*

Proof. Let a function p from $H^1_{\Gamma_1}(\Omega_1,\rho_1)$ satisfy problem II for $\tau_1 \in H_{\Gamma_1}^{\frac{1}{2}}$. Let the field $\vec{v}$ belong to $\vec{G}_{h,S_1}(\Omega_1,\rho_1)$. By virtue of the conditions $\operatorname{div}\vec{v} = 0$ (in Ω_1) and $\vec{v}\cdot\vec{n}_1 = 0$ (on S_1), we see that

$$(\rho_1^{-1}(x_3)\nabla p, \vec{v})_{\vec{G}_{h,S_1}(\Omega_1,\rho_1)} = \int\limits_{\Omega_1} \nabla p\cdot\overline{\vec{v}}\, d\Omega_1 = -\int\limits_{\Omega_1} p\,\overline{\operatorname{div}\vec{v}}\, d\Omega_1$$
$$+ \int\limits_{\partial\Omega_1} p\,\overline{(\vec{v}\cdot\vec{n}_1)}\, dS = \int\limits_{\Gamma_1} p\,\overline{\gamma_1\vec{v}}\, d\Gamma_1 = \int\limits_{\Gamma_1} \rho_1(0)\tau_1\overline{\gamma_1\vec{v}}\, d\Gamma_1$$

or

$$(G_1\tau_1, \vec{v})_{\vec{G}_{h,S_1}(\Omega_1,\rho_1)} = (\tau_1, \gamma_1\vec{v})_{H_1}, \qquad \forall\tau_1 \in H_{\Gamma_1}^{\frac{1}{2}} \quad \forall\vec{v}\in\vec{G}_{h,S_1}(\Omega_1,\rho_1). \tag{1.58}$$

Identity (1.58) is the definition of the adjoint operator. Thus, $\gamma_1^* = G_1$. Identity (1.58) can be extended on $\vec{v} \in \vec{J}_{0,S_1}(\Omega_1,\rho_1)$ because elements of $\vec{J}_0(\Omega_1,\rho_1)$ are orthogonal to elements of $\vec{G}_{h,S_1}(\Omega_1,\rho_1)$ and $\vec{v}\cdot\vec{n}_1 = 0$ (on $\partial\Omega_1$) for $\vec{v}\in\vec{J}_0(\Omega_1,\rho_1)$. Thus, $\gamma_1 : \vec{J}_{0,S_1}(\Omega_1,\rho_1) \to H_{\Gamma_1}^{-\frac{1}{2}}$, while $G_1 : H_{\Gamma_1}^{\frac{1}{2}} \to \vec{G}_{h,S_1}(\Omega_1,\rho_1) \subset \vec{J}_{0,S_1}(\Omega_1,\rho_1)$. This completes the proof. □

The following lemma can be proved in the same way.

Lemma 1.7. *The operators γ_2 and G_2 are adjoint to each other, $\gamma_2\colon \vec{G}_2(\Omega_2,\rho_2) \to H_{\Gamma_2}^{-\frac{1}{2}}$, and $G_2\colon H_{\Gamma_2}^{\frac{1}{2}} \to \vec{G}_2(\Omega_2,\rho_2)$.*

Lemma 1.8. *The operator K from* (1.54) *is bounded.*

Proof. This assertion follows from the boundedness of all operator coefficients K_i $(i = \overline{1,4})$ of the matrix K. For example, let us check the boundedness of the operator $K_2 = -\rho_1^{-1}(0)\rho_2(0)G_1C_3\gamma_2$. It acts from $\vec{G}_2(\Omega_2,\rho_2)$ to $\vec{J}_{0,S_1}(\Omega_1,\rho_1)$. Take an arbitrary field $\vec{w}_2 \in \vec{G}_2(\Omega_2,\rho_2)$. By virtue of the definition of the space $\vec{G}_2(\Omega_2,\rho_2)$, we have $\gamma_2\vec{w}_2 \in H_{\Gamma_2}^{-\frac{1}{2}}$. To find the function Ψ_2 for $\eta_2 = \gamma_2\vec{w}_2$ (on Γ_2), we use auxiliary problem IV, which is uniquely solvable. Further, we have

$$C_3\gamma_2\vec{w}_2 = C_3\eta_2 = \rho_2^{-1}(0)P_{\Gamma_1}\Psi_2|_{\Gamma_1} \in H_{\Gamma_1}^{\frac{1}{2}}.$$

Finally, the operator considered acts as follows (up to a constant factor):

$$G_1C_3\gamma_2\vec{w}_2 = G_1C_3\eta_2 = G_1P_{\Gamma_1}(\Psi_2|_{\Gamma_1}) =: \vec{u}_1 \in \vec{G}_{h,S_1}(\Omega_1,\rho_1).$$

All operators mentioned above are bounded. Hence, the operator K_2 is bounded. The remaining coefficients K_i are of a similar type. Thus, we have proved the boundedness of the operator K. This completes the proof of the lemma. $\square$

Lemma 1.9. *The operator K is self-adjoint in $\vec{J}_{0,S}(\Omega,\rho)$.*

Proof. Using the definitions of the operators K_i, properties of solutions of auxiliary problems III and IV, the decompositions for $\vec{u}_2$ and Φ_2, and the fact that the operators γ_k and G_k are adjoint $(k=1,2)$, we directly find the quadratic form of the operator K in the complex Hilbert space $\vec{J}_{0,S}(\Omega,\rho)$:

$$(K\vec{u},\vec{u})_{\vec{J}_{0,S}(\Omega,\rho)} = \int\limits_{\Omega_1} \rho_1(x_3)|\vec{u}_1|^2\,d\Omega_1 + \int\limits_{\Omega_2} \rho_2^{-1}(x_3)|\nabla\Phi_2|^2\,d\Omega_2 + \int\limits_{\Omega_2} \rho_2(x_3)|\vec{v}_2|^2\,d\Omega_2. \tag{1.59}$$

Taking into account Lemma 1.8, from (1.59), we see that the operator K is self-adjoint. This completes the proof. $\square$

Lemma 1.10. *The operator K is positive definite.*

Proof. First, consider the operator D from (1.55) and prove its boundedness. Taking into account the decomposition

$$\vec{u}_1 = \vec{w}_{1,1} + \rho_1^{-1}(x_3)\nabla w_{1,2}, \quad \vec{w}_{1,1}\in\vec{J}_0(\Omega_1,\rho_1), \quad \rho_1^{-1}(x_3)\nabla w_{1,2}\in\vec{G}_{h,S_1}(\Omega_1,\rho_1), \tag{1.60}$$

and properties of auxiliary problem V, we have

$$\begin{aligned}
\|D\vec{w}_1\|^2_{\vec{G}_1(\Omega_2,\rho_2)} &= \int\limits_{\Omega_2} \rho_2^{-1}(x_3)|\nabla\Psi_1|^2\,d\Omega_2 = \int\limits_{\Omega_2} \nabla\Psi_1\cdot\overline{(\rho_2^{-1}(x_3)\nabla\Psi_1)}\,d\Omega_2\\
&= \int\limits_{\partial\Omega_2} \Psi_1\cdot\overline{(\rho_2^{-1}\nabla\Psi_1\cdot\vec{n}_2)}\,dS = \int\limits_{\Gamma_1} P_{\Gamma_1}\Psi_1\cdot\overline{(\rho_2^{-1}(0)\nabla\Psi_1\cdot\vec{n}_2)}\,d\Gamma_1\\
&= \int\limits_{\Gamma_1} \rho_2(0)C_1(\rho_2^{-1}(0)\nabla\Psi_1\cdot\vec{n}_2)\overline{(\rho_2^{-1}(0)\nabla\Psi_1\cdot\vec{n}_2)}\,d\Gamma_1 = \|C_1^{\frac12}(\rho_2^{-1}(0)\nabla\Psi_1\cdot\vec{n}_2)\|^2_{H_1}\\
&\le d\|C_{0,4}^{\frac12}(\rho_2^{-1}(0)\nabla\Psi_1\cdot\vec{n}_2)\|^2_{H_1} = d\|C_{0,4}^{\frac12}(-\rho_1^{-1}(0)\nabla w_{1,2}\cdot\vec{n}_1)\|^2_{H_1}\\
&= d\int\limits_{\Gamma_1} \rho_1(0)C_{0,4}(\rho_1^{-1}(0)\nabla w_{1,2}\cdot\vec{n}_1)(\rho_1^{-1}(0)\nabla w_{1,2}\cdot\vec{n}_1)\,d\Gamma_1\\
&= d\int\limits_{\Gamma_1} P_{\Gamma_1}w_{1,2}\overline{(\rho_1^{-1}(0)\nabla w_{1,2}\cdot\vec{n}_1)}\,d\Gamma_1 = d\int\limits_{\Omega_1} \rho_1^{-1}(x_3)|\nabla w_{1,2}|^2\,d\Omega_1\\
&\le d\int\limits_{\Omega_1} \rho_1^{-1}(x_3)|\nabla w_{1,2}|^2\,d\Omega_1 + d\int\limits_{\Omega_1} \rho_1(x_3)|\vec{w}_{1,1}|^2\,d\Omega_1\\
&= d\int\limits_{\Omega_1} \rho_1(x_3)|\vec{w}_1|^2\,d\Omega_1 = d\|\vec{w}_1\|^2_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)}
\end{aligned}$$

for any $\vec{u}_1\in\vec{J}_{0,S_1}(\Omega_1,\rho_1)$, where d is the positive constant from the equivalence relation for the norms from (1.33).

Now we prove that the operator K is positive definite. Using quadratic form (1.59), we have

$$(K\vec{u},\vec{u})_{\vec{J}_{0,S}(\Omega,\rho)} = \int\limits_{\Omega_1} \rho_1(x_3)|\vec{u}_1|^2\,d\Omega_1 + \int\limits_{\Omega_2} \rho_2^{-1}(x_3)|\nabla\Phi_2|^2\,d\Omega_2 + \int\limits_{\Omega_2} \rho_2(x_3)|\vec{v}_2|^2\,d\Omega_2$$
$$= \|\vec{u}_1\|^2_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)} + \|D\vec{u}_1\|^2_{\vec{G}_1(\Omega_2,\rho_2)} + \|\vec{w}_2\|^2_{\vec{G}_2(\Omega_2,\rho_2)} + 2\mathrm{Re}(D\vec{u}_1,\vec{w}_2)_{\vec{G}_{\widehat{h,S_2}}(\Omega_2,\rho_2)}$$
$$+ \|\vec{v}_2\|^2_{\vec{J}_0(\Omega_2,\rho_2)} \geq \|\vec{u}_1\|^2_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)} + \|D\vec{u}_1\|^2_{\vec{G}_1(\Omega_2,\rho_2)} + \|\vec{w}_2\|^2_{\vec{G}_2(\Omega_2,\rho_2)}$$
$$- \left(\varepsilon\|D\vec{u}_1\|^2_{\vec{G}_1(\Omega_2,\rho_2)} + \frac{1}{\varepsilon}\|\vec{w}_2\|^2_{\vec{G}_2(\Omega_2,\rho_2)}\right) + \|\vec{v}_2\|^2_{\vec{J}_0(\Omega_2,\rho_2)} \geq \|\vec{u}_1\|^2_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)}$$
$$+ \big(1-\varepsilon\big)\|D\|^2\|\vec{u}_1\|^2_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)} + \left(1-\frac{1}{\varepsilon}\right)\|\vec{w}_2\|^2_{\vec{G}_2(\Omega_2,\rho_2)} + \|\vec{v}_2\|^2_{\vec{J}_0(\Omega_2,\rho_2)}$$
$$= \left(1+(1-\varepsilon)\|D\|^2\right)\|\vec{u}_1\|^2_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)} + \left(1-\frac{1}{\varepsilon}\right)\|\vec{w}_2\|^2_{\vec{G}_2(\Omega_2,\rho_2)} + \|\vec{v}_2\|^2_{\vec{J}_0(\Omega_2,\rho_2)}$$

for any $\vec{u} \in \vec{J}_{0,S}(\Omega,\rho)$.

The latter inequality is valid under the assumption that $\varepsilon > 1$. Let us show that ε can be selected to satisfy the positive definiteness condition for the operator K, i.e., to satisfy the equations

$$\varepsilon > 1, \quad 1 + (1-\varepsilon)\|D\|^2 > 0.$$

This implies the inequality $1 < \varepsilon < 1 + 1/\|D\|^2$; hence, one can assign $\varepsilon := 1 + \Theta/\|D\|^2$, where $0 < \Theta < 1$.

Thus, fixing such a Θ, we obtain an $\varepsilon > 1$ such that the constants at the second powers of the norms are strictly positive. Selecting the least of those constants and denoting it by $C(\Theta)$, we get the condition of positive definiteness for the operator K. This completes the proof. □

Lemma 1.11. *The operators* $E_1 : \mathfrak{L}_2(\Omega) \to \vec{J}_{0,S}(\Omega,\rho)$ *and* $E_2 : \vec{J}_{0,S}(\Omega,\rho) \to \mathfrak{L}_2(\Omega)$ *are such that* $-E_1^* = E_2$, *where*

$$\|E_1\| = \|E_2\| \leq N, \quad N := N_{0,1} + N_{0,2}d + 2N_{0,2}, \tag{1.61}$$

and $d > 0$ *is the constant from the equivalence relation for the norms from* (1.33).

Proof. To prove that $E_2 = -E_1^*$, we use the definitions of the operators E_1 and E_2, properties of auxiliary problems III and IV, decompositions (1.42) for Φ_2, and the fact that the operators γ_k and G_k are adjoint to each other, $k = 1, 2$.

The proof of inequality (1.61) follows directly from Lemma 1.2, Lemma 1.3, and the inequality

$$|(E_1\rho,\vec{u})| \leq \big(N_{0,1} + N_{0,2}d + 2N_{0,2}\big)\|\vec{u}\|_{\vec{J}_{0,S}(\Omega,\rho)}\|\rho\|_{\mathfrak{L}_2(\Omega)}.$$

This completes the proof. □

Remark 1.4. The operators F_1 and F_2 are such that $F_1 = -F_2^*$ (see their definitions). The operator F_1 is defined on the set $\oplus_{i=1}^2 H_{\Gamma_i}^{\frac{1}{2}}$, which is dense in H. It is unbounded as an operator from H to $\vec{J}_{0,S}(\Omega,\rho)$. The operator F_2 is unbounded as well.

1.8. On strong solutions of the Cauchy problem. For convenience, we write system (1.52) in the orthogonal sum of Hilbert spaces

$$\widetilde{H}^{(1)} := \vec{J}_{0,S_1}(\Omega_1,\rho_1) \oplus H_1 \oplus \mathfrak{L}_2(\Omega_1) \oplus H^{(1)}$$

as follows:

$$
\begin{pmatrix} K_1 & 0 & 0 & \widetilde{K_2} \\ 0 & g\rho_1^{-1}(0)\Delta\rho_1 I_{\Gamma_1} & 0 & 0 \\ 0 & 0 & I_{\mathfrak{L}_2(\Omega_1)} & 0 \\ \widetilde{K_3} & 0 & 0 & M_1 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \vec{u}_1 \\ \zeta_1 \\ \widetilde{\rho}_1 \\ v \end{pmatrix}
+ \begin{pmatrix} \mu A & g\rho_1^{-1}(0)\Delta\rho_1 G_1 & \widetilde{C}_1 & L_1 \\ -g\rho_1^{-1}(0)\Delta\rho_1\gamma_1 & 0 & 0 & 0 \\ -\widetilde{C}_1^* & 0 & 0 & 0 \\ L_2 & 0 & 0 & M_2 \end{pmatrix} \begin{pmatrix} \vec{u}_1 \\ \zeta_1 \\ \widetilde{\rho}_1 \\ v \end{pmatrix} = \begin{pmatrix} f^{(1)} \\ 0 \\ 0 \\ f^{(2)} \end{pmatrix}, \quad (1.62)
$$

where

$$
H^{(1)} := \vec{G}_2(\Omega_2, \rho_2) \oplus \vec{J}_0(\Omega_2, \rho_2) \oplus H_2 \oplus \mathfrak{L}_2(\Omega_2), \quad (\vec{u}_1; \zeta_1; \widetilde{\rho}_1; v)^t := (\vec{u}_1; \zeta_1; \widetilde{\rho}_1; \vec{w}_2, \vec{v}_2, \zeta_2, \widetilde{\rho}_2)^t,
$$

$$
\widetilde{K}_2 = (K_2, 0, 0, 0), \quad \widetilde{K}_3 = (K_3, 0, 0, 0)^t,
$$

$$
M_1 = \begin{pmatrix} K_4 & 0 & 0 & 0 \\ 0 & I_0 & 0 & 0 \\ 0 & 0 & gI_{\Gamma_2} & 0 \\ 0 & 0 & 0 & I_{\mathfrak{L}_2(\Omega_2)} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & gG_2 & \rho_2^{-1}(b)G_2 P_{\Gamma_2} B \\ 0 & 0 & 0 & \widetilde{C}_{2,0} \\ -g\gamma_2 & 0 & 0 & 0 \\ -\widetilde{C}_{2,1}^* & -\widetilde{C}_{2,0}^* & 0 & 0 \end{pmatrix}, \quad (1.63)
$$

$$
f^{(1)} = P_{0,S_1}\vec{f} - \rho_1^{-1}(0)G_1 P_{\Gamma_1} F, \quad f^{(2)} = (\rho_2^{-1}(b)G_2 P_{\Gamma_2} F; P_{0,1}\vec{f}; 0; 0)^t.
$$

The operators L_1 and L_2 are such that $L_2 = -L_1^*$ (it follows from Lemma 1.11); they have the following form:

$$
L_1 = (0, 0, 0, -\rho_1^{-1}(0)G_1 P_{\Gamma_1} B); \quad L_2 = (0, 0, 0, -\widetilde{C}_{2,1}^* D)^t. \quad (1.64)
$$

The initial-value data for Eq. (1.62) takes the form

$$
((\vec{u}_1; \zeta_1; \widetilde{\rho}_1; v)(0))^t = (P_{0,S_1}\vec{u}_1^0; \zeta_1^0; \widetilde{\rho}_1^0; v^0)^t, \quad v^0 = (\Pi_2 P_{\widehat{h,S_2}}\vec{u}_2^0, P_0\vec{u}_2^0, \zeta_2^0, \widetilde{\rho}_2^0). \quad (1.65)
$$

Introduce the following notation:

$$
y := (\vec{u}_1; \zeta_1; \widetilde{\rho}_1; v)^t, \quad f := (f^{(1)}; 0; 0; f^{(2)})^t,
$$

$$
\mathcal{R} := \begin{pmatrix} K_1 & 0 & 0 & \widetilde{K_2} \\ 0 & g\rho_1^{-1}(0)\Delta\rho_1 I_{\Gamma_1} & 0 & 0 \\ 0 & 0 & I_{\mathfrak{L}_2(\Omega_1)} & 0 \\ \widetilde{K_3} & 0 & 0 & M_1 \end{pmatrix}, \quad \mathcal{P} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I_{\Gamma_1} & 0 & 0 \\ 0 & 0 & I_{\mathfrak{L}_2(\Omega_1)} & 0 \\ 0 & 0 & 0 & I \end{pmatrix},
$$

$$
\mathcal{A}_0 := \begin{pmatrix} \mu A & g\rho_1^{-1}(0)\Delta\rho_1 G_1 & \widetilde{C}_1 & L_1 \\ -g\rho_1^{-1}(0)\Delta\rho_1\gamma_1 & 0 & 0 & 0 \\ -\widetilde{C}_1^* & 0 & 0 & 0 \\ -L_1^* & 0 & 0 & M_2 \end{pmatrix},
$$

where

$$
\mathcal{D}(\mathcal{A}_0) = \mathcal{D}(A) \oplus \mathcal{D}(G_1) \oplus \mathfrak{L}_2(\Omega_1) \oplus \mathcal{D}(M_2), \quad (1.66)
$$

while I is the identity operator in $H^{(1)}$. Then Cauchy problem (1.62), (1.65) takes the form

$$
\mathcal{R}\frac{dy}{dt} + \mathcal{A}_0 y = f, \quad y(0) = y^0. \quad (1.67)
$$

Lemma 1.12. *The following assertions are true*:

(1) $0 \ll \mathcal{R} \in \mathcal{L}(\widetilde{H}^{(1)})$, $\quad 0 \ll M_1 \in \mathcal{L}(H^{(1)})$;

(2) *the operator* M_2 *is unbounded in* $H^{(1)}$ *and* $M_2^* = -M_2$.

The proof of those assertions follow from the lemmas and considerations of Sec. 1.7.

Definition 1.1. Functions $\vec{u}_i$, ζ_i, $\widetilde{\rho}_i$, and p_i $(i = 1, 2)$ are called a *strong solution of problem* (1.2)–(1.5) if the vector $y(t) = (\vec{u}_1(t); \zeta_1(t); \widetilde{\rho}_1(t); v(t))^t$ is a strong solution of Cauchy problem (1.67) and trivial relations (1.24), (1.25), and (1.19) are satisfied in the sense of distributions. A function $y(t)$ is called a *strong solution of Cauchy problem* (1.67) if $y(t) \in \mathcal{D}(\mathcal{A}_0)$ for any t from $[0, T]$, $\mathcal{A}_0 y(t) \in C([0, T]; \widetilde{H}^{(1)})$, $y(t) \in C^1([0, T]; \widetilde{H}^{(1)})$, and the equation and initial-value condition from (1.67) are satisfied for any t from $[0, T]$.

To prove the existence theorem for a strong solution of Cauchy problem (1.67), perform the following transformations.

In Eq. (1.67), we change the sought function: $y(t) = e^t y_1(t)$. This yields the following equation with respect to y_1:

$$\mathcal{R}\frac{dy_1}{dt} + (\mathcal{A}_0 + \varepsilon\mathcal{P})y_1 + (\mathcal{R} - \varepsilon\mathcal{P})y_1 = e^{-t} f, \tag{1.68}$$

where a positive ε is such that $\mathcal{R} - \varepsilon\mathcal{P} \gg 0$. It is possible to select such an ε because $\mathcal{R} \gg 0$ in $\widetilde{H}^{(1)}$.

The domain $\mathcal{D}(\mathcal{A}_0 + \varepsilon\mathcal{P})$ of the operator $\mathcal{A}_0 + \varepsilon\mathcal{P}$ coincides with $\mathcal{D}(\mathcal{A}_0)$ from (1.66). The operator $\mathcal{A}_0 + \varepsilon\mathcal{P}$ is uniformly is accretive on $\mathcal{D}(\mathcal{A}_0)$, i.e.,

$$\operatorname{Re}(\mathcal{A}_0 + \varepsilon\mathcal{P}) = \operatorname{diag}(\mu A, \varepsilon I_{\Gamma_1}, \varepsilon I_{\mathfrak{L}_2(\Omega_1)}, \varepsilon I) \gg 0 \quad \text{on} \quad \mathcal{D}(\mathcal{A}_0).$$

However, it is not closed because the operator γ_1 is unbounded in $\vec{J}_{0,S_1}(\Omega_1, \rho_1)$ and $\mathcal{D}(\gamma_1) \supset \mathcal{D}(A)$. Thus, the operator $\mathcal{A}_0 + \varepsilon\mathcal{P}$ is not maximal accretive.

Introduce the following notation:

$$g\rho_1^{-1}(0)\Delta\rho_1\gamma_1 A^{-\frac{1}{2}} =: Q_1, \quad g\rho_1^{-1}(0)\Delta\rho_1 A^{-\frac{1}{2}} G_1 =: Q_1^+, \quad \mathcal{D}(Q_1^+) := \mathcal{D}(G_1).$$

The following lemma takes place.

Lemma 1.13. *$Q_1^+ \subset Q_1^*$, $Q_1^+ = Q_1^*|_{\mathcal{D}(G_1)}$, and $\overline{Q_1^+} = Q_1^*$.*

Proof. Let $\vec{u}_1 \in \vec{J}_{0,S_1}(\Omega_1, \rho_1)$ and $\zeta_1 \in \mathcal{D}(G_1)$. Then

$$\begin{aligned}(Q_1\vec{u}_1, \zeta_1)_{H_1} &= (g\rho_1^{-1}(0)\Delta\rho_1\gamma_1 A^{-\frac{1}{2}}\vec{u}_1, \zeta_1)_{H_1} = (g\rho_1^{-1}(0)\Delta\rho_1 A^{-\frac{1}{2}}\vec{u}_1, G_1\zeta_1)_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)} \\ &= (\vec{u}_1, g\rho_1^{-1}(0)\Delta\rho_1 A^{-\frac{1}{2}} G_1\zeta_1)_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)} = (\vec{u}_1, Q_1^+\zeta_1)_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)}.\end{aligned}$$

This implies that $Q_1^+ \subset Q_1^*$ and $Q_1^+ = Q_1^*|_{\mathcal{D}(G_1)}$. Let us prove that the operator Q_1^+ is bounded on $\mathcal{D}(G_1)$. The operator Q_1 is a bounded operator from $\vec{J}_{0,S_1}(\Omega_1, \rho_1)$ to H_1. In fact, the operator $A^{-\frac{1}{2}}$ maps $\vec{J}_{0,S_1}(\Omega_1, \rho_1)$ onto $\vec{J}^1_{0,S_1}(\Omega_1, \rho_1)$, while γ_1 is a compact operator from $\vec{J}^1_{0,S_1}(\Omega_1, \rho_1)$ in H_1 due to the trace theorem. Thus, the operator Q_1 is bounded; moreover, it is compact. Therefore, the operator Q_1^* is bounded as well. Then the following inequality holds for any $\zeta_1 \in \mathcal{D}(G_1)$:

$$\|Q_1^+\zeta_1\|_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)} = \|Q_1^*\zeta_1\|_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)} \le \|Q_1^*\|\|\zeta_1\|_{H_1}.$$

Hence, the operator Q_1^+ is bounded on $\mathcal{D}(G_1)$. Therefore, it can be extended, by continuity, till the bounded operator Q_1^*, i.e., $\overline{Q_1^+} = Q_1^*$. □

The operator $\mathcal{A}_0 + \varepsilon\mathcal{P}$ admits closure (see [13, p. 109]). Denote its closure by $\mathcal{A}$. Then the relation

$$\mathcal{A} + (\mathcal{R} - \varepsilon\mathcal{P}) = \overline{(\mathcal{A}_0 + \varepsilon\mathcal{P})} + (\mathcal{R} - \varepsilon\mathcal{P}) = \overline{(\mathcal{A}_0 + \varepsilon\mathcal{P}) + (\mathcal{R} - \varepsilon\mathcal{P})}$$

holds because the operator $\mathcal{R} - \varepsilon\mathcal{P}$ is bounded and defined on the whole space. Taking this into account, we close only the operator $\mathcal{A}_0 + \varepsilon\mathcal{P}$.

Lemma 1.14. *The closure* $\mathcal{A} := \overline{\mathcal{A}_0 + \varepsilon\mathcal{P}}$ *of the operator* $\mathcal{A}_0 + \varepsilon\mathcal{P}$ *is a maximal accretive operator,*

$$\mathcal{D}(\mathcal{A}) = \{(\vec{u}_1, \zeta_1, \widetilde{\rho}_1, v)^t \quad | \quad \mu\vec{u}_1 + A^{-\frac{1}{2}}Q_1^*\zeta_1 \in \mathcal{D}(A), \quad v \in \mathcal{D}(M_2)\},$$

and

$$\mathcal{A} = \mathcal{T}_A \mathcal{Q}_A \mathcal{T}_A,$$

where

$$\mathcal{T}_A := \begin{pmatrix} A^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & I_{\Gamma_1} & 0 & 0 \\ 0 & 0 & I_{\mathfrak{L}_2(\Omega_1)} & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad \mathcal{Q}_A := \begin{pmatrix} \mu I_1 & Q_1^* & Q_2^* & Q_3^* \\ -Q_1 & \varepsilon I_{\Gamma_1} & 0 & 0 \\ -Q_2 & 0 & \varepsilon I_{\mathfrak{L}_2(\Omega_1)} & 0 \\ -Q_3 & 0 & 0 & \varepsilon I + M_2 \end{pmatrix},$$

and

$$Q_2 := \widetilde{C}_1^* A^{-\frac{1}{2}} \quad Q_3 := L_1^* A^{-\frac{1}{2}}.$$

Proof. It is easy to check that the operator $\mathcal{A}_0 + \varepsilon\mathcal{P}$ can be represented as $\mathcal{T}_A \mathcal{Q}_A^+ \mathcal{T}_A$, where

$$\mathcal{Q}_A^+ := \begin{pmatrix} \mu I_1 & Q_1^+ & Q_2^* & Q_3^* \\ -Q_1 & \varepsilon I_{\Gamma_1} & 0 & 0 \\ -Q_2 & 0 & \varepsilon I_{\mathfrak{L}_2(\Omega_1)} & 0 \\ -Q_3 & 0 & 0 & \varepsilon I + M_2 \end{pmatrix}.$$

To close the operator $\mathcal{A}_0 + \varepsilon\mathcal{P}$ is to replace Q_1^+ by Q_1^* in its middle block. In fact, such a closure represents the operator $\mathcal{A}$ as the following product of closed operators: $\mathcal{A} = \mathcal{T}_1\, \mathcal{T}_2\, \mathcal{T}_1$. Moreover, $\mathcal{T}_1^{-1} \in \mathcal{L}(\widetilde{H}^{(1)})$ because the operator $A^{-\frac{1}{2}}$ is bounded in $\vec{J}_{0,S_1}(\Omega_1, \rho_1)$. Then we directly check that all elements of the inverse matrix $\mathcal{T}_2^{-1}$ are bounded operators. Hence, $\mathcal{T}_2^{-1} \in \mathcal{L}(\widetilde{H}^{(1)})$ and the operator $\mathcal{A}$ is closed.

Consider the domain $\mathcal{D}(\mathcal{A})$ of the operator $\mathcal{A}$. First, it follows from the representation for the operator $\mathcal{A}$ that $\vec{u}_1 \in \mathcal{D}(A^{\frac{1}{2}})$ and $v \in \mathcal{D}(M_2)$. Further, the following expression should have a sense:

$$\begin{pmatrix} A^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & I_{\Gamma_1} & 0 & 0 \\ 0 & 0 & I_{\mathfrak{L}_2(\Omega)} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} \mu A^{\frac{1}{2}}\vec{u}_1 + Q_1^*\zeta_1 + Q_2^*\widetilde{\rho}_1 + Q_3^* v \\ -Q_1 A^{\frac{1}{2}}\vec{u}_1 + \varepsilon\zeta_1 \\ -Q_2 A^{\frac{1}{2}}\vec{u}_1 + \varepsilon\widetilde{\rho}_1 \\ -Q_3 A^{\frac{1}{2}}\vec{u}_1 + (\varepsilon I + M_2)v \end{pmatrix};$$

this means that $\mu A^{\frac{1}{2}}\vec{u}_1 + Q_1^*\zeta_1 + Q_2^*\widetilde{\rho}_1 + Q_3^* v \in \mathcal{D}(A^{\frac{1}{2}})$ or

$$\mu\vec{u}_1 + A^{-\frac{1}{2}}Q_1^*\zeta_1 + A^{-\frac{1}{2}}Q_2^*\widetilde{\rho}_1 + A^{-\frac{1}{2}}Q_3^* v = \mu\vec{u}_1 + A^{-\frac{1}{2}}Q_1^*\zeta_1 + A^{-1}\widetilde{C}_1\widetilde{\rho}_1 + A^{-1}L_1 v \in \mathcal{D}(A).$$

Thus, we conclude that

$$\mathcal{D}(\mathcal{A}) = \{\, (\vec{u}_1, \zeta_1, \tilde{\rho}_1, v)^t \quad | \quad \mu\vec{u}_1 + A^{-\frac{1}{2}}Q_1^*\zeta_1 \in \mathcal{D}(A), \quad v \in \mathcal{D}(M_2)\,\}.$$

Note that the condition $\vec{u}_1 \in \mathcal{D}(A^{\frac{1}{2}})$ follows from the condition $\mu\vec{u}_1 + A^{-\frac{1}{2}}Q_1^*\zeta_1 \in \mathcal{D}(A)$. In fact, since $\mathcal{D}(A) \subset \mathcal{D}(A^{\frac{1}{2}})$ and $A^{-\frac{1}{2}}Q_1^*\zeta_1 \in \mathcal{D}(A^{\frac{1}{2}})$ for any $\zeta_1 \in H_1$, it follows that $\vec{u}_1$ belongs to $\mathcal{D}(A^{\frac{1}{2}})$ as well. This completes the proof. □

Instead of (1.68), consider the equation with the closed operator:

$$\mathcal{R}\frac{dy_1}{dt} + \mathcal{A}y_1 + (\mathcal{R} - \varepsilon\mathcal{P})y_1 = e^{-t}f. \tag{1.69}$$

The operator $\mathcal{A} + \mathcal{R} - \varepsilon\mathcal{P}$ is maximal accretive. The operator $\mathcal{R}$ is self-adjoint, positive definite, and bounded in $\widetilde{H}^{(1)}$. Hence, there exists an operator $\mathcal{R}^{-1}$ possessing the same properties. Transform Eq. (1.69) to the following form:

$$\frac{dy_1}{dt} = -\mathcal{R}^{-1}(\mathcal{A} + \mathcal{R} - \varepsilon\mathcal{P})y_1 + \mathcal{R}^{-1}e^{-t}f, \quad y_1(0) = y^0. \tag{1.70}$$

Introduce an equivalent norm in $\widetilde{H}^{(1)}$ as follows:

$$\langle y_1, y_1\rangle := (\mathcal{R}y_1, y_1)_{\widetilde{H}^{(1)}} = (\mathcal{R}^{\frac{1}{2}}y_1, \mathcal{R}^{\frac{1}{2}}y_1)_{\widetilde{H}^{(1)}}.$$

Its equivalence to the previous norm follows from the properties of the operator $\mathcal{R}$.

It is easy to check that the operator $-\mathcal{R}^{-1}(\mathcal{A}+\mathcal{R}-\varepsilon\mathcal{P})$ is maximal dissipative with respect to the new scalar product. Hence, Cauchy problem (1.70) is uniformly correct (see [13, p. 166]), while the operator $-\mathcal{R}^{-1}(\mathcal{A}+\mathcal{R}-\varepsilon\mathcal{P})$ generates the semigroup of operators $\mathcal{U}(t) := \exp\left(-t\mathcal{R}^{-1}(\mathcal{A}+\mathcal{R}-\varepsilon\mathcal{P})\right)$. This semigroup is contracting with respect to the new scalar product. Therefore, the following theorem holds.

Theorem 1.3. *Cauchy problem* (1.70) *has a unique strong solution on* $[0,T]$ *represented as*

$$y_1(t) = \mathcal{U}(t)y^0 + \int_0^t \mathcal{U}(t-\tau)\mathcal{R}^{-1}e^{-\tau}f(\tau)\,d\tau$$

if the following conditions are satisfied: $y^0 \in \mathcal{D}(\mathcal{A})$, $f(t) \in C^1([0,T];\widetilde{H}^{(1)})$.

Thus, the Cauchy problem

$$\mathcal{R}\frac{dy_1}{dt} + \mathcal{A}y_1 + (\mathcal{R}-\varepsilon\mathcal{P})y_1 = e^{-t}f, \quad y_1(0) = y^0$$

has a unique strong solution if $y^0 \in \mathcal{D}(\mathcal{A})$ and $f(t) \in C^1(\,[0,T];\widetilde{H}^{(1)}\,)$; in particular, it has a unique strong solution if $y^0 \in \mathcal{D}(\mathcal{A}_0)$.

Denoting $y_1 =: (\vec{u}_{11};\zeta_{11};\tilde{\rho}_{11};v_1)^t$, we see that Eq. (1.69) may be represented as the following system:

$$\begin{cases} K_1\dfrac{d\vec{u}_{11}}{dt} + \widetilde{K}_2\dfrac{dv_1}{dt} + A(\mu\vec{u}_{11} + A^{-\frac{1}{2}}Q_1^*\zeta_{11}) \\ \qquad\qquad + \widetilde{C}_1\tilde{\rho}_{11} + L_1v_1 + K_1\vec{u}_{11} + \widetilde{K}_2v_1 = e^{-t}f^{(1)}, \\ g\rho_1^{-1}(0)\Delta\rho_1\dfrac{d\zeta_{11}}{dt} - Q_1A^{\frac{1}{2}}\vec{u}_{11} + \varepsilon\zeta_{11} + (g\rho_1^{-1}(0)\Delta\rho_1 - \varepsilon)\zeta_{11} = 0, \\ \dfrac{d\tilde{\rho}_{11}}{dt} - Q_2A^{\frac{1}{2}}\vec{u}_{11} + \varepsilon\tilde{\rho}_{11} + (1-\varepsilon)\tilde{\rho}_{11} = 0, \\ \widetilde{K}_3\dfrac{d\vec{u}_{11}}{dt} + M_1\dfrac{dv_1}{dt} - Q_3A^{\frac{1}{2}}\vec{u}_{11} + (\varepsilon I + M_2)v_1 + (M_1 - \varepsilon I)v_1 = e^{-t}f^{(2)}, \end{cases} \tag{1.71}$$

$$(\vec{u}_{11},\zeta_{11},\tilde{\rho}_{11},v_1)^t(0) = (\vec{u}_1,\zeta_1,\tilde{\rho}_1,v)^{0t}. \tag{1.72}$$

Note that one cannot open the brackets in the first equation yet because any term inside the brackets might belong to $\mathcal{D}(A^{\frac{1}{2}})$, while only their sum belongs to $\mathcal{D}(A)$.

To get back from problem (1.70) to problem (1.68), multiply the second equation of system (1.71) by e^t. Then

$$g\rho_1^{-1}(0)\Delta\rho_1\frac{d}{dt}(e^t\zeta_{11}) = e^tQ_1A^{\frac{1}{2}}\vec{u}_{11}.$$

This implies that $\zeta_{11}(t)$ is expressed as follows:

$$\zeta_{11}(t) = \frac{1}{g\rho_1^{-1}(0)\Delta\rho_1}\int_0^t e^{-(t-s)}Q_1A^{\frac{1}{2}}\vec{u}_{11}(s)\,ds + e^{-t}\zeta_1^0. \tag{1.73}$$

Substituting function (1.73) in the first equation of system (1.71), we obtain

$$\begin{aligned} K_1\frac{d\vec{u}_{11}}{dt} + \widetilde{K}_2\frac{dv_1}{dt} + A\biggl(\mu\vec{u}_{11} + e^{-t}A^{-\frac{1}{2}}Q_1^*\zeta_1^0 + \frac{1}{g\rho_1^{-1}(0)\Delta\rho_1}A^{-\frac{1}{2}}Q_1^*\int_0^t e^{-(t-s)}Q_1A^{\frac{1}{2}}\vec{u}_{11}(s)\,ds\biggr) \\ + \widetilde{C}_1\tilde{\rho}_{11} + L_1v_1 + \widetilde{K}_2v_1 = e^{-t}f^{(1)}. \end{aligned} \tag{1.74}$$

It follows from Theorem 1.3 that the function

$$\mu\vec{u}_{11} + e^{-t}A^{-\frac{1}{2}}Q_1^*\zeta_1^0 + \frac{1}{g\rho_1^{-1}(0)\Delta\rho_1}A^{-\frac{1}{2}}Q_1^*\int_0^t e^{-(t-s)}Q_1A^{\frac{1}{2}}\vec{u}_{11}(s)\,ds =: \vec{v}^1(t) \tag{1.75}$$

belongs to $\mathcal{D}(A)$ for any $t \in [0,T]$ and $\vec{v}^1(t) \in C([0,T];\mathcal{D}(A))$. If $\zeta_1^0 \in \mathcal{D}(G_1)$, then

$$Q_1^*\zeta_1^0 = Q_1^+\zeta_1^0 = g\rho_1^{-1}(0)\Delta\rho_1 A^{-\frac{1}{2}}G_1\zeta_1^0$$

(due to Lemma 1.13). Therefore, we have

$$A^{-\frac{1}{2}}Q_1^*\zeta_1^0 = g\rho_1^{-1}(0)\Delta\rho_1 A^{-1}G_1\zeta_1^0 \in \mathcal{D}(A).$$

Consider the operator

$$\mathfrak{K} := A^{-\frac{1}{2}}Q_1^*Q_1A^{\frac{1}{2}} = g\rho_1^{-1}(0)\Delta\rho_1 A^{-\frac{1}{2}}Q_1^*\gamma_1.$$

Assume that $\mathcal{D}(A)$ is a Hilbert space with the graph norm:

$$\|\vec{v}\|_{\mathcal{D}(A)} := \|A\vec{v}\|. \tag{1.76}$$

Then the restriction $\mathfrak{K}^1 := \mathfrak{K}|_{\mathcal{D}(A)}$ is a linear bounded operator in $\mathcal{D}(A)$. In fact, if $\vec{u}_{11} \in \mathcal{D}(A) \subset \mathcal{D}(A^{\frac{1}{2}}) = \vec{J}^1_{0,S_1}(\Omega_1,\rho_1)$, then $\gamma_1\vec{u}_{11} \in H^{\frac{1}{2}}_{\Gamma_1} = \mathcal{D}(G_1)$. From Lemma 1.13, we have

$$\mathfrak{K}^1\vec{u}_{11} = g\rho_1^{-1}(0)\Delta\rho_1 A^{-\frac{1}{2}}Q_1^*\gamma_1\vec{u}_{11} = g\rho_1^{-1}(0)\Delta\rho_1 A^{-\frac{1}{2}}Q_1^+\gamma_1\vec{u}_{11} = (g\rho_1^{-1}(0)\Delta\rho_1)^2A^{-1}G_1\gamma_1\vec{u}_{11} \in \mathcal{D}(A).$$

Since $G_1\gamma_1$ is a bounded operator from $\vec{J}^1_{0,S_1}(\Omega_1,\rho_1) = \mathcal{D}(A^{\frac{1}{2}})$ to $\vec{J}_{0,S_1}(\Omega_1,\rho_1)$, it follows that $\mathfrak{K}^1 : \mathcal{D}(A) \to \mathcal{D}(A)$ is a bounded operator. The fact proved allows us to treat relation (1.75) as a second-kind Volterra equation in the space $\mathcal{D}(A)$ (with graph norm (1.76)). The function $\vec{v}^1(t) - e^{-t}A^{-\frac{1}{2}}Q_1^*\zeta_1^0$ belongs to $C([0,T];\mathcal{D}(A))$, while the kernel $\mathfrak{K}^1e^{-(t-s)}$ of the integral operator is continuous in $\mathcal{D}(A)$ with respect to (t,s). Hence, Eq. (1.75) has a unique solution $\vec{u}_{11} \in C([0,T];\mathcal{D}(A))$ and any term of Eq. (1.75) belongs to $C([0,T];\mathcal{D}(A))$. Thus, we can open the brackets in Eq. (1.74) and in the first equation of (1.71). This yields that Eq. (1.68) holds for the function $y_1(t) = (\vec{u}_{11}, \zeta_{11}, \tilde{\rho}_{11}, v_1)^t$. Applying the inverse change of variables $y_1(t) = e^{-t}y(t)$ to Eq. (1.68), we obtain the following theorem for the Cauchy problem

$$\mathcal{R}\frac{dy}{dt} + \mathcal{A}_0 y = f, \quad y(0) = y^0. \tag{1.77}$$

Theorem 1.4. *Cauchy problem* (1.77) *has a unique strong solution on* $[0,T]$ *if the following conditions are satisfied*:

$$y^0 \in \mathcal{D}(\mathcal{A}_0), \quad f(t) \in C^1([0,T];\widetilde{H}^{(1)}).$$

Theorem 1.4 contains the result about the existence and uniqueness of a strong solution for the Cauchy problem (1.77). For problem (1.2)–(1.5), this result is modified as follows:

Theorem 1.5. *Initial-boundary value problem* (1.2)–(1.5) *has a unique strong solution on* $[0,T]$ *if the following conditions are satisfied*:

(1) $\vec{u}_1^0 \in \mathcal{D}(A)$, $\vec{u}_2^0 \in \vec{H}^1(\Omega_2,\rho_2) \cap \vec{J}_{0,\widehat{S_2}}(\Omega_2, rho_2)$, $\zeta_i^0 \in H^{\frac{1}{2}}_{\Gamma_i}$, $\tilde{\rho}_i^0 \in \mathfrak{L}_2(\Omega_i)$ $(i = 1,2)$, *and* $\gamma_1P_{0,S_1}\vec{u}_1^0(x) = -\widetilde{\gamma}_2\Pi_1P_{\widehat{h,S_2}}\vec{u}_2^0(x)$ $(on\ \Gamma_1)$;

(2) $\vec{f}(t) \in C^1(\,[0,T];\vec{L}_2(\Omega,\rho)\,)$ $(\Omega = \Omega_1 \cup \Omega_2)$.

2. Problem on Normal Oscillations

2.1. Problem posing. The problem on small motions of the system described in Sec. 1.1 is reduced to Cauchy problem (1.71)–(1.72) with a closed operator. Consider normal oscillations of the given hydraulic system, i.e., solutions of system (1.71) (or (1.69)) with $\vec{f} \equiv \vec{0}$ such that

$$y_1(t) = \exp(-\lambda t)y = \exp(-\lambda t)(\vec{u}_1; \zeta_1; \widetilde{\rho}_1; v)^t. \tag{2.1}$$

This leads to the following spectral problem:

$$\begin{cases} (1-\lambda)K_1\vec{u}_1 + (1-\lambda)K_2\vec{w}_2 + A(\mu\vec{u}_1 + A^{-\frac{1}{2}}Q_1^*\zeta_1) \\ \qquad\qquad + \widetilde{C}_1\widetilde{\rho}_1 - \rho_1^{-1}(0)G_1P_{\Gamma_1}B\widetilde{\rho}_2 = 0, \\ g\rho_1^{-1}(0)\Delta\rho_1(1-\lambda)\zeta_1 - Q_1A^{\frac{1}{2}}\vec{u}_1 = 0, \\ (1-\lambda)\widetilde{\rho}_1 - Q_2A^{\frac{1}{2}}\vec{u}_1 = 0, \\ (1-\lambda)K_3\vec{u}_1 + (1-\lambda)K_4\vec{w}_2 + gG\zeta_2 + \rho_2^{-1}(b)G_2P_{\Gamma_2}B\widetilde{\rho}_2 = 0, \\ (1-\lambda)\vec{v}_2 + \tilde{C}_{2,0}\widetilde{\rho}_2 = 0, \\ (1-\lambda)g\zeta_2 - g\gamma_2\vec{w}_2 = 0, \\ (1-\lambda)\widetilde{\rho}_2 - \widetilde{C}_{2,1}^*\vec{w}_2 - \widetilde{C}_{2,0}^*\vec{v}_2 - \widetilde{C}_{2,1}^*D\vec{u}_1 = 0. \end{cases} \tag{2.2}$$

Lemma 2.1. *The operator $\widehat{B}_2^{-1} := (G_2\gamma_2)^{-1}$ is self-adjoint, positive, and compact in the Hilbert space $\vec{G}_2(\Omega_2, \rho_2)$.*

Proof. Consider the eigenvalue problem for the operator ${\gamma_2}^{-1}{G_2}^{-1}$:

$${\gamma_2}^{-1}{G_2}^{-1}(\rho_2^{-1}\nabla\Psi_2) = \lambda(\rho_2^{-1}\nabla\Psi_2), \quad \rho_2^{-1}\nabla\Psi_2 \in \vec{G}_2(\Omega_2, \rho_2).$$

Apply the operator γ_2 to both sides of the equation. Taking into account that ${G_2}^{-1}(\rho_2^{-1}\nabla\Psi_2) = \rho_2^{-1}(b)\Psi_2|_{\Gamma_2}$, we obtain the relation

$$\rho_2^{-1}(b)\Psi_2|_{\Gamma_2} = \lambda\gamma_2(\rho_2^{-1}\nabla\Psi_2) = \lambda(\rho_2^{-1}\nabla\psi_2 \cdot \vec{n}_2)|_{\Gamma_2}.$$

This can be written as

$$C_4\eta_2 = \lambda\eta_2 \tag{2.3}$$

(we use the relation $C_4\eta_2 = \rho_2^{-1}(b)\Psi_2|_{\Gamma_2}$).

Thus, the eigenvalue problem for the operator ${\gamma_2}^{-1}{G_2}^{-1}$ is equivalent to problem (2.3), which is the eigenvalue problem for the operator C_4. The latter operator is self-adjoint, positive, and compact in L_{2,Γ_2}. Thus, the spectrum of problem (2.3) (as well as the eigenvalue problem for the operator ${\gamma_2}^{-1}{G_2}^{-1}$) is a sequence of positive eigenvalues of finite multiplicities with the only limit point at the origin (since the operator C_4 is infinite dimensional). This implies that the operator ${\gamma_2}^{-1}{G_2}^{-1}$ is compact. This completes the proof. ☐

Consider the main case where $\lambda \neq 1$ (note that $\lambda = 1$ is an eigenvalue of infinite multiplicity and the set of relative quiescent states corresponds to that eigenvalue).

System (2.2) contains unbounded operators. To reduce it to a system with bounded operator coefficients, we express ζ_1, $\widetilde{\rho}_1$, ζ_2, $\widetilde{\rho}_2$ from the second, third, sixth, and seventh equations of system (2.2) and substitute them in the first, fourth, and fifth equations. Then we change the variables in system (2.2): $A^{\frac{1}{2}}\vec{u}_1 = \vec{\psi}_1$ ($\vec{\psi}_1 \in \vec{J}_{0,S_1}(\Omega_1, \rho_1)$). Further, we apply the bounded (and compact) operator $A^{-\frac{1}{2}}$ to its first equation. In system (2.2), we change the variables: $\widehat{B}_2^{\frac{1}{2}}\vec{w}_2 = \vec{\psi}_2$ (it follows from Lemma 2.1 that the operator $\widehat{B}_2$ is positive definite and unbounded); then we apply the operator

$\widehat{B}_2^{-\frac{1}{2}}$ to the second equation and multiply all equations by -1. We obtain

$$\begin{cases} (\lambda-1)A_{1,1}\vec{\psi}_1 + (\lambda-1)A_{1,2}\vec{\psi}_2 - \mu I_1\vec{\psi}_1 + (\lambda-1)^{-1}g\widehat{B}_1\vec{\psi}_1 \\ \qquad\qquad + (\lambda-1)^{-1}\big(T_{1,1}\vec{\psi}_1 + T_{1,2}\vec{\psi}_2 + T_{1,3}\vec{v}_2\big) = 0, \\ (\lambda-1)A_{2,1}\vec{\psi}_1 + (\lambda-1)A_{2,2}\vec{\psi}_2 + (\lambda-1)^{-1}gI_2\vec{\psi}_2 \\ \qquad\qquad + (\lambda-1)^{-1}\big(T_{2,1}\vec{\psi}_1 + T_{2,2}\vec{\psi}_2 + T_{2,3}\vec{v}_2\big) = 0, \\ (\lambda-1)I_0\vec{v}_2 + (\lambda-1)^{-1}\big(T_{3,1}\vec{\psi}_1 + T_{3,2}\vec{\psi}_2 + C_v\vec{v}_2\big) = 0, \end{cases} \tag{2.4}$$

where

$$\begin{array}{lll} A_{1,1} := A^{-\frac{1}{2}}K_1A^{-\frac{1}{2}}, & A_{1,2} := A^{-\frac{1}{2}}K_2\widehat{B}_2^{-\frac{1}{2}}, & A_{2,1} := \widehat{B}_2^{-\frac{1}{2}}K_3A^{-\frac{1}{2}}, \\ A_{2,2} := \widehat{B}_2^{-\frac{1}{2}}K_4\widehat{B}_2^{-\frac{1}{2}}, & \widehat{B}_1 = \rho_1^{-1}(0)\Delta\rho_1\Theta_1^*\Theta_1, & A_c := A^{-\frac{1}{2}}\widetilde{C}_1\widetilde{C}_1^*A^{-\frac{1}{2}}, \\ S_1 := -\rho_1^{-1}(0)A^{-\frac{1}{2}}G_1P_{\Gamma_1}B, & S_2 := \rho_2^{-1}(b)G_2P_{\Gamma_2}B, & \Theta_1 := \gamma_1A^{-\frac{1}{2}}, \\ T_{1,1} := A_c + S_1\widetilde{C}_{2,1}^*DA^{-\frac{1}{2}}, & T_{1,2} := S_1\widetilde{C}_{2,1}^*\widetilde{B}_2^{-\frac{1}{2}}, & T_{1,3} := S_1\widetilde{C}_{2,0}^*, \\ T_{2,1} := \widehat{B}_2^{-\frac{1}{2}}S_2\widetilde{C}_{2,1}^*DA^{-\frac{1}{2}}, & T_{2,2} := \widehat{B}_2^{-\frac{1}{2}}S_2\widetilde{C}_{2,1}^*\widehat{B}_2^{-\frac{1}{2}}, & T_{2,3} := \widehat{B}_2^{-\frac{1}{2}}S_2\widetilde{C}_{2,0}^*, \\ T_{3,1} := \widetilde{C}_{2,0}\widetilde{C}_{2,1}^*DA^{-\frac{1}{2}}, & T_{3,2} := \widetilde{C}_{2,0}\widetilde{C}_{2,1}^*\widehat{B}_2^{-\frac{1}{2}}, & C_v := \widetilde{C}_{2,0}\widetilde{C}_{2,0}^*, \end{array}$$

and I_1, I_2, and I_0 are the identity operators in the Hilbert spaces $\vec{J}_{0,S_1}(\Omega_1,\rho_1)$, $\vec{G}_2(\Omega_2,\rho_2)$, and $\vec{J}_0(\Omega_2,\rho_2)$, respectively.

The operators $A_{i,j}$ $(i,j=1,2)$ and $T_{r,n}$ $(r,n=\overline{1,3})$ are compact because the operators $B_2^{-\frac{1}{2}}$ and $A^{-\frac{1}{2}}$ are compact, while the operators K_i $(i=\overline{1,4})$, $\widetilde{C}_1$, $\widetilde{C}_1^*$, $\widetilde{C}_{2,1}^*$, $\widetilde{C}_{2,0}^*$, $\widetilde{C}_{2,0}$, D, S_1, and S_2 are bounded. Consider the operator $\widehat{B}_1 = \rho_1^{-1}(0)\Delta\rho_1(\gamma_1A^{-\frac{1}{2}})^*(\gamma_1A^{-\frac{1}{2}})$. The operator $A^{-\frac{1}{2}}$ maps $\vec{J}_{0,S_1}(\Omega_1,\rho_1)$ onto $\vec{J}^1_{0,S_1}(\Omega_1,\rho_1)$. By virtue of the trace theorem, γ_1 is a compact operator from $\vec{J}^1_{0,S_1}(\Omega_1,\rho_1)$ to L_{2,Γ_1} (see [8]). Therefore, the product $\Theta_1 = \gamma_1A^{-\frac{1}{2}}$ is a compact operator from $\vec{J}_{0,S_1}(\Omega_1,\rho_1)$ to L_{2,Γ_1}. We obtain that $\widehat{B}_1$ is a compact nonnegative operator.

Lemma 2.2. *The operator C_v is self-adjoint*; *if* (1.1) *is satisfied, then its limit spectrum coincides with the segment* $[-iN_{0,2};iN_{0,2}]$.

The proof is given in [10].

Change the spectral parameter in system (2.4): $\lambda-1=\widehat{\lambda}$. Denote $\widehat{\lambda}$ by λ again. Multiply the equations by λ and write the obtained system as a single vector-matrix equation in the orthogonal sum of the Hilbert spaces $\vec{J}_{0,S}(\Omega,\rho)$:

$$\lambda^2\begin{pmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & A_{2,2} & 0 \\ 0 & 0 & I_0 \end{pmatrix}\begin{pmatrix}\vec{\psi}_1\\ \vec{\psi}_2\\ \vec{v}_2\end{pmatrix} - \lambda\mu\begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix}\vec{\psi}_1\\ \vec{\psi}_2\\ \vec{v}_2\end{pmatrix}$$
$$+\, g\begin{pmatrix} \widehat{B}_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix}\vec{\psi}_1\\ \vec{\psi}_2\\ \vec{v}_2\end{pmatrix} + \begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} \\ T_{2,1} & T_{2,2} & T_{2,3} \\ T_{3,1} & T_{3,2} & C_v \end{pmatrix}\begin{pmatrix}\vec{\psi}_1\\ \vec{\psi}_2\\ \vec{v}_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}, \tag{2.5}$$

where $(\vec{\psi}_1;\vec{\psi}_2;\vec{v}_2)^t \in \vec{J}_{0,S}(\Omega,\rho) = \vec{J}_{0,S_1}(\Omega_1,\rho_1)\oplus\vec{G}_2(\Omega_2,\rho_2)\oplus\vec{J}_0(\Omega_2,\rho_2)$.

The link between eigenelements and associated elements of spectral problems (2.2) and (2.5) is given by the following assertion.

Theorem 2.1. *Let $\lambda_0 \neq 1$ be an eigenvalue of spectral problem* (2.2) *such that its multiplicity is finite. Let $y_0, y_1, \ldots, y_k$ be the corresponding chain of the eigenelement and the associated elements, $y_j = (\vec{u}_1^j; \zeta_1^j; \widetilde{\rho}_1^j; \vec{w}_2^j; \vec{v}_2^j; \zeta_2^j; \widetilde{\rho}_2^j)^t$ $(j = \overline{0,k})$. Then the elements $z_0, z_1, \ldots, z_k$,*

$$z_j = (A^{\frac{1}{2}}\vec{u}_1^j; \widehat{B}_2^{\frac{1}{2}}\vec{w}_2^j; \vec{v}_2^j)^t,$$

form a chain of the eigenelement and the corresponding associated elements of spectral problem (2.5) *and this chain corresponds to the eigenvalue $\lambda = \lambda_0 - 1$.*

Conversely, for any chain of eigenelements and associated elements of spectral problem (2.5) *corresponding to an eigenvalue $\lambda_0 - 1 \neq 0$ of finite multiplicity there exists a chain of root elements $y_0, y_1, \ldots, y_k$ of spectral problem* (2.2) *corresponding to the eigenvalue $\lambda = \lambda_0$. For the elements y_j $(j = \overline{0,k})$, we have*

$$y_j = (A^{-\frac{1}{2}}\vec{\psi}_1^j; \zeta_1^j; \widetilde{\rho}_1^j; \widehat{B}_2^{-\frac{1}{2}}\vec{\psi}_2^j; \vec{v}_2^j; \zeta_2^j; \widetilde{\rho}_2^j)^t,$$

$$\zeta_1^j = \sum_{i=0}^{j}(1-\lambda_0)^{i-j-1}\Theta_1\vec{\psi}_1^i, \qquad \widetilde{\rho}_1^j = \sum_{i=0}^{j}(1-\lambda)^{i-j-1}Q_2\psi_1^i,$$

$$\zeta_2^j = \sum_{i=0}^{j}(1-\lambda)^{i-j-1}\gamma_2\widehat{B}_2^{-\frac{1}{2}}\vec{\psi}_2^i,$$

$$\widetilde{\rho}_2^j = \sum_{i=0}^{j}(1-\lambda)^{i-j-1}\left(\widetilde{C}_{2,1}^*\widehat{B}_2^{-\frac{1}{2}}\vec{w}_2^i + \widetilde{C}_{2,0}^*\vec{v}_2^i + \widetilde{C}_{2,1}^*DA^{-\frac{1}{2}}\vec{\psi}_1^i\right) \qquad (j = \overline{0,k}).$$

To prove this theorem, we use the scheme described in [2].

2.2. On the essential spectrum and the localization of the spectrum. To reduce problem (2.5) to the spectral problem for a Fredholm operator-function, we write Eq. (2.5) as a system; then we divide its first equation by $-\lambda\mu$ (assuming that $\lambda \neq 0$), divide its second equation by g, and apply the operator-function $C_{v,0}(\lambda) := (\lambda^2 I_0 + C_v)^{-1}$ to its third equation. This yields

$$(\widetilde{I} + \widetilde{F}(\lambda))z = 0, \tag{2.6}$$

where $\widetilde{I}$ is the identity operator in $\vec{J}_{0,S}(\Omega)$, while the operator-function $\widetilde{F}(\lambda)$ is defined as follows:

$$\widetilde{F}(\lambda) := \begin{pmatrix} -\frac{\lambda}{\mu}A_{1,1} - \frac{g}{\lambda\mu}\widehat{B}_1 - \frac{1}{\lambda\mu}T_{1,1} & -\frac{\lambda}{\mu}A_{1,2} - \frac{1}{\lambda\mu}T_{1,2} & -\frac{1}{\lambda\mu}T_{1,3} \\ \frac{\lambda^2}{g}A_{2,1} + \frac{1}{g}T_{2,1} & \frac{\lambda^2}{g}A_{2,2} + \frac{1}{g}T_{2,2} & \frac{1}{g}T_{2,3} \\ C_{v,0}(\lambda)T_{3,1} & C_{v,0}(\lambda)T_{3,2} & 0 \end{pmatrix}. \tag{2.7}$$

It follows from the structure of $\widetilde{F}(\lambda)$ that the operator pencil of problem (2.6) is a Fredholm operator-function. Let us prove that if $\lambda = -\alpha$ $(\alpha > 0)$, then the pencil is invertible and the inverse is bounded as well. It follows from (2.5) and (2.6) that

$$\widetilde{I} + \widetilde{F}(\lambda) = \begin{pmatrix} (-\lambda\mu)^{-1} & 0 & 0 \\ 0 & g^{-1}I_2 & 0 \\ 0 & 0 & C_{v,0}(\lambda) \end{pmatrix} L(\lambda). \tag{2.8}$$

The fact that the pencil $L(\lambda)$ is invertible and the inverse is bounded if $\lambda = -\alpha$ $(\alpha > 0)$ is checked directly. Thus, it follows from (2.8) that the operator pencil of problem (2.6) has a regular value. Since the operator-function $\widetilde{F}(\lambda)$ loses its analyticity at the infinity, origin, and the points of the continuous spectrum of operator C_v (see (2.7) and Lemma 2.2), it follows that the pencil of problem (2.6) is regular in $\mathbb{C} \setminus [-iN_{0,2}, iN_{0,2}]$. This implies (see [8, p. 74]) that all points of the spectrum located outside the segment $[-iN_{0,2}, iN_{0,2}]$ are eigenvalues of the operator-function of problem (2.6); hence, they are eigenvalues of $L(\lambda)$. The multiplicity of any corresponding eigenelement is finite. Only singularities of the operator-function, i.e., the segment $[-iN_{0,2}, iN_{0,2}]$, and ∞ can be condensation points of the spectrum.

Lemma 2.3.

(1) *The nonzero point spectrum of the pencil* $L(\lambda)$ *is located in the right-hand closed half-plane* (*i.e.*, $\operatorname{Re}\lambda \geq 0$) *and is symmetric with respect to the real axis.*

(2) *The limit spectrum of the pencil* $L(\lambda)$ *coincides with the segment* $[-iN_{0,2}, iN_{0,2}]$.

This lemma is proved in [19].

2.3. Asymptotic properties of branches of eigenvalues with the limit point at infinity. Change the variables in Eq. (2.5):

$$\begin{pmatrix} I_1 & 0 & 0 \\ Q & I_2 & 0 \\ 0 & 0 & I_0 \end{pmatrix} \begin{pmatrix} \vec{\psi}_1 \\ \vec{\psi}_2 \\ \vec{v}_2 \end{pmatrix} =: \begin{pmatrix} \vec{\psi}_1 \\ \vec{\varphi} \\ \vec{v}_2 \end{pmatrix}, \tag{2.9}$$

$$Q := A_{2,2}^{-1} A_{2,1}. \tag{2.10}$$

Then apply the operator

$$\begin{pmatrix} I_1 & -Q^* & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_0 \end{pmatrix}$$

from the left to that equation. We obtain the spectral problem

$$\lambda^2 \begin{pmatrix} T & 0 & 0 \\ 0 & A_{2,2} & 0 \\ 0 & 0 & I_0 \end{pmatrix} \begin{pmatrix} \vec{\psi}_1 \\ \vec{\varphi} \\ \vec{v}_2 \end{pmatrix} - \lambda\mu \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{\psi}_1 \\ \vec{\varphi} \\ \vec{v}_2 \end{pmatrix} \tag{2.11}$$

$$+ g \begin{pmatrix} B_Q & -Q^* & 0 \\ -Q & I_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{\psi}_1 \\ \vec{\varphi} \\ \vec{v}_2 \end{pmatrix} + \begin{pmatrix} \widetilde{T}_{1,1} & \widetilde{T}_{1,2} & \widetilde{T}_{1,3} \\ \widetilde{T}_{2,1} & T_{2,2} & T_{2,3} \\ \widetilde{T}_{3,1} & T_{3,2} & C_v \end{pmatrix} \begin{pmatrix} \vec{\psi}_1 \\ \vec{\varphi} \\ \vec{v}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$(\vec{\psi}_1; \vec{\varphi}; \vec{v}_2)^t \in \vec{J}_{0,S}(\Omega, \rho) = \vec{J}_{0,S_1}(\Omega_1, \rho_1) \oplus \vec{G}_2(\Omega_2, \rho_2) \oplus \vec{J}_0(\Omega_2, \rho_2),$$

where $T := A_{1,1} - A_{1,2}A_{2,2}^{-1}A_{2,1}$, $\widetilde{T}_{1,1} := T_{1,1} - Q^*T_{2,1} - T_{1,2}Q + Q^*T_{2,2}Q$, $\widetilde{T}_{1,2} := T_{1,2} - Q^*T_{2,2}$, $\widetilde{T}_{1,3} := T_{1,3} - Q^*T_{2,3}$, $\widetilde{T}_{2,1} := T_{2,1} - T_{2,2}Q$, $\widetilde{T}_{3,1} := T_{3,1} - T_{3,2}Q$, and $B_Q := \widehat{B}_1 + Q^*Q$.

Lemma 2.4. *The following assertions are true*:

$$0 < T \in \mathfrak{S}_\infty, \quad A_{1,2}A_{2,2}^{-\frac{1}{2}} \in \mathfrak{S}_\infty, \quad A_{2,2}^{-\frac{1}{2}}A_{2,1} \in \mathfrak{S}_\infty.$$

Proof. Using the notation of Eqs. (2.4) and (2.11), transform the following expression:

$$T = A_{1,1} - A_{1,2}A_{2,2}^{-1}A_{2,1} = A^{-\frac{1}{2}}\left(K_1 - K_2K_4^{-1}K_3\right)A^{-\frac{1}{2}}.$$

Since the operator $A^{-\frac{1}{2}}$ is positive and compact in $\vec{J}_{0,S_1}(\Omega_1, \rho_1)$, it is sufficient to show that

$$0 \ll K_1 - K_2K_4^{-1}K_3 \in \mathfrak{L}(\vec{J}_{0,S_1}(\Omega_1, \rho_1))$$

in order to prove the first assertion of the lemma.

As we have proved above, the operator block $\begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$ is positive definite and bounded in $\vec{J}_{0,S_1}(\Omega_1, \rho_1) \oplus \vec{G}_2(\Omega_2, \rho_2)$. In terms of its quadratic form, the positive definiteness condition means that

$$\left(\begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}\begin{pmatrix} \vec{u}_1 \\ \vec{w}_2 \end{pmatrix}, \begin{pmatrix} \vec{u}_1 \\ \vec{w}_2 \end{pmatrix}\right)_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)\oplus\vec{G}_2(\Omega_2,\rho_2)} = (K_1\vec{u}_1, \vec{w}_2)_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)}$$
$$+ (K_2\vec{w}_2, \vec{u}_1)_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)} + (K_3\vec{u}_1, \vec{w}_2)_{\vec{G}_2(\Omega_2,\rho_2)} + (K_4\vec{w}_2, \vec{w}_2)_{\vec{G}_2(\Omega_2,\rho_2)}$$
$$\geq \text{const}\left(\|\vec{u}_1\|^2_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)} + \|\vec{w}_2\|^2_{\vec{G}_2(\Omega_2,\rho_2)}\right)$$

for any $\vec{u}_1 \in \vec{J}_{0,S_1}(\Omega_1, \rho_1)$ and $\vec{w}_2 \in \vec{G}_2(\Omega_2, \rho_2)$. Change the variables: $\vec{w}_2 = -K_4^{-1}K_3\vec{u}_1$. Then we have

$$((K_1 - K_2K_4^{-1}K_3)\vec{u}_1, \vec{u}_1)_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)} \geq \text{const}\|\vec{u}_1\|^2_{\vec{J}_{0,S_1}(\Omega_1,\rho_1)}$$

for any $\vec{u}_1 \in \vec{J}_{0,S_1}(\Omega_1, \rho_1)$.

This implies the positive definiteness of the operator $K_1 - K_2K_4^{-1}K_3$. Thus, the first assertion is proved.

It follows from the first assertion that the operator $A_{1,2}A_{2,2}^{-1}A_{2,1}$ is compact. Moreover, it is self-adjoint and positive. The operator $A_{1,2}A_{2,2}^{-1}A_{2,1}$ can be represented as follows:

$$A_{1,2}A_{2,2}^{-1}A_{2,1} = (A_{1,2}A_{2,2}^{-\frac{1}{2}})(A_{2,2}^{-\frac{1}{2}}A_{2,1}).$$

This implies that the operators $A_{1,2}A_{2,2}^{-\frac{1}{2}}$ and $A_{2,2}^{-\frac{1}{2}}A_{2,1}$ are compact as well. This completes the proof. □

Lemma 2.5. *Eigenvalues of the operator $A_{2,2}$ asymptotically behave as follows*:

$$\lambda_k(A_{2,2}) = 2g\Big(\frac{\pi}{\operatorname{mes}\Gamma}\Big)^{\frac{1}{2}}k^{\frac{1}{2}}[1 + o(1)] \quad (k \to \infty). \tag{2.12}$$

Proof. The fact that the eigenvalue problem for the operator $A_{2,2}$ is equivalent to the eigenvalue problem for the operator C_4 is checked directly. The asymptotic relation for eigenvalues of the operator C_4 is obtained in [10]. □

Introduce the notation $C_{v,0}(\lambda) := (\lambda^2 I_0 + C_v)^{-1}$. Assume that $\lambda \notin [-iN_{0,2}, iN_{0,2}]$. Write problem (2.11) as a system of equations. Taking into account the assumption about λ, from the third equation of the system, we have

$$\vec{v}_2 = -C_{v,0}(\lambda)\widetilde{T}_{3,1}\vec{\psi}_1 - C_{v,0}(\lambda)T_{3,2}\vec{\varphi}. \tag{2.13}$$

Substituting (2.13) in the two remaining equations, we obtain the following problem in vector-matrix form:

$$\lambda^2\begin{pmatrix} T & 0 \\ 0 & A_{2,2} \end{pmatrix}\begin{pmatrix} \vec{\psi}_1 \\ \vec{\varphi} \end{pmatrix} - \lambda\mu\begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \vec{\psi}_1 \\ \vec{\varphi} \end{pmatrix} + g\begin{pmatrix} B_Q & -Q^* \\ -Q & I_2 \end{pmatrix}\begin{pmatrix} \vec{\psi}_1 \\ \vec{\varphi} \end{pmatrix}$$
$$+ \begin{pmatrix} R_{1,1}(\lambda) & R_{1,2}(\lambda) \\ R_{2,1}(\lambda) & R_{2,2}(\lambda) \end{pmatrix}\begin{pmatrix} \vec{\psi}_1 \\ \vec{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.14}$$

where $R_{i,j}(\lambda) := \widetilde{T}_{i,j} + \widetilde{T}_{i,3}C_{v,0}(\lambda)\widetilde{T}_{3,j}$, $i, j = 1, 2$ (if there is no unity in the subscript of the operator $\widetilde{T}_{i,j}$, then the tilde over it should be taken off).

We need the following auxiliary result.

Lemma 2.6. *Let $0 < C = C^* \in \mathfrak{S}_\infty$ and the eigenvalues of the operator C asymptotically behave as a power. Introduce the notation*:

$$l(\lambda) := I + \lambda^2 C + G(\lambda), \quad T(\lambda) := (I - \lambda C^{\frac{1}{2}})^{-1}G(\lambda)(I + \lambda C^{\frac{1}{2}})^{-1},$$
$$\Lambda^{\pm}_{R,\varepsilon} := \{\,\lambda \mid |\lambda| > R,\ |\arg\lambda \mp \pi/2| < \varepsilon\,\}, \quad -\pi < \arg\lambda \leq \pi.$$

Let the operator-function $G(\lambda)$ be analytic in the sectors $\Lambda^{+}_{R,\varepsilon}$ and $\Lambda^{-}_{R,\varepsilon}$, while the operator-function $T(\lambda)$ satisfy the following condition: $T(\lambda) \to 0 \;\; (\lambda \to \infty,\;\; \lambda \in \Lambda^{\pm}_{R,\varepsilon})$. *Then*

$$\lambda_k^{\pm i} = \pm i \lambda_k^{\frac{1}{2}}(C^{-1})(1 + o(1)) \quad (k \to \infty).$$

The proof is given in [16].

Theorem 2.2. *For any (small) positive ε and sufficiently large $R = R(\varepsilon) > N_{0,2}$, spectral problem* (2.11) *has two complex conjugate branches $\{\lambda_k^{\pm i}\}_{k=1}^{\infty}$ of eigenvalues located in the sectors $\Lambda^{+}_{R,\varepsilon}$ and $\Lambda^{-}_{R,\varepsilon}$; they asymptotically behave as follows*:

$$\lambda_k^{\pm i} = \pm i g^{\frac{1}{2}} \lambda_k^{\frac{1}{2}}(A_{2,2}^{-1})(1 + o(1)) \quad (k \to \infty). \tag{2.15}$$

Proof. Represent matrix pencil (2.14) as a system and reduce the first equation to the following form:

$$K(\lambda)\vec{\psi}_1 := \left(I_1 - \frac{\lambda}{\mu}T - \frac{g}{\lambda\mu}B_Q - \frac{1}{\lambda\mu}R_{1,1}(\lambda)\right)\vec{\psi}_1 = \frac{1}{\lambda\mu}\left(-gQ^* + R_{1,2}(\lambda)\right)\vec{\varphi}. \tag{2.16}$$

The pencil $K(\lambda)$ is a pencil of the S. Kreĭn type. The structure of its spectrum is known. For any positive ε, we can select $R = R(\varepsilon)$ such that $K(\lambda)$ is continuously invertible in $\Lambda^{\pm}_{R,\varepsilon}$. Assume that $\lambda \in \Lambda^{\pm}_{R,\varepsilon}$ and express $\vec{\psi}_1$ from relation (2.16):

$$\vec{\psi}_1 = -\frac{1}{\lambda\mu}K^{-1}(\lambda)\left(-gQ^* + R_{1,2}(\lambda)\right)\vec{\varphi}.$$

Substitute this expression in the second equation of pencil (2.14) and write it as follows:

$$l_1(\lambda)\vec{\varphi} := \left(I_2 + \frac{\lambda^2}{g}A_{2,2} + G_1(\lambda)\right)\vec{\varphi} = 0, \quad \vec{\varphi} \in \vec{G}_2(\Omega_2, \rho_2), \tag{2.17}$$

$$G_1(\lambda) := \frac{1}{g}R_{2,2}(\lambda) - \frac{1}{\lambda\mu}QK^{-1}(\lambda)\left(-gQ^* + R_{1,2}(\lambda)\right) + \frac{g}{\lambda\mu}R_{2,1}(\lambda)K^{-1}(\lambda)\left(-gQ^* + R_{1,2}(\lambda)\right). \tag{2.18}$$

To satisfy the conditions of Lemma 2.6, we have to prove that the operator-function

$$T(\lambda) := (I_2 - \lambda g^{-\frac{1}{2}}A_{2,2}^{\frac{1}{2}})^{-1}G_1(\lambda)(I_2 + \lambda g^{-\frac{1}{2}}A_{2,2}^{\frac{1}{2}})^{-1} \tag{2.19}$$

satisfies the following condition: $T(\lambda) \to 0 \;\; (\lambda \to \infty, \;\; \lambda \in \Lambda^{\pm}_{R,\varepsilon})$. It suffices to prove that the operator-function

$$K^{-1}(\lambda) = \left(I_1 - (I_1 - \frac{\lambda}{\mu}T)^{-1}\frac{1}{\lambda\mu}(R_{1,1}(\lambda) + gB_Q)\right)^{-1}\left(I_1 - \frac{\lambda}{\mu}T\right)^{-1}$$

is bounded in the domain $\Lambda^{\pm}_{R,\varepsilon}$ and the first term in (2.18), (2.19) behaves at least as $o(1)$ as $\lambda \to \infty$, $\lambda \in \Lambda^{\pm}_{R,\varepsilon}$.

It is known that the norm of the resolvent of the self-adjoint operator T is such that

$$\|(T - \lambda I)^{-1}\| = (\rho(\lambda, \sigma(T)))^{-1}, \quad \rho(\lambda, \sigma(T)) := \min_{z \in \sigma(T)} |\lambda - z|. \tag{2.20}$$

Then we have

$$\left\|(I_1 - \frac{\lambda}{\mu}T)^{-1}\right\| = \left\|\frac{1}{\lambda}(\frac{1}{\lambda}I_1 - \frac{1}{\mu}T)^{-1}\right\| = \frac{1}{|\lambda|\rho\left(\frac{1}{\lambda}, \sigma(\frac{T}{\mu})\right)} \leq \frac{1}{\cos\varepsilon},$$

$$\left\|(I_1 - \frac{\lambda}{\mu}T)^{-1}\frac{1}{\lambda\mu}(R_{1,1}(\lambda) + gB_Q)\right\| \leq \frac{\|R_{1,1}(\lambda)\| + g\|B_Q\|}{|\lambda|\mu\cos\varepsilon} \to 0,$$

$$\left\|\left(I_1 - (I_1 - \frac{\lambda}{\mu}T)^{-1}\frac{1}{\lambda\mu}(R_{1,1}(\lambda) + gB_Q)\right)^{-1}\right\| \to 1 \quad (\lambda \to \infty, \quad \lambda \in \Lambda^{\pm}_{R,\varepsilon})$$

for any $\lambda \in \Lambda^{\pm}_{R,\varepsilon}$.

Thus, the following inequality holds in the specified domain:

$$\|K^{-1}(\lambda)\| \le (1+\delta)(\cos\varepsilon)^{-1} \quad \forall\,\delta > 0 \quad (\lambda \to \infty).$$

Using the definition of $R_{2,2}(\lambda)$ (it is defined after (2.14)), we estimate the first term of (2.18), (2.19) for $\lambda \to \infty,\ \lambda \in \Lambda^{\pm}_{R,\varepsilon}$:

$$\left\|(I_2 - \lambda g^{-\frac{1}{2}} A^{\frac{1}{2}}_{2,2})^{-1}\frac{1}{g}R_{2,2}(\lambda)(I_2 + \lambda g^{-\frac{1}{2}} A^{\frac{1}{2}}_{2,2})^{-1}\right\| \le \left\|(g^{\frac{1}{2}} I_2 - \lambda A^{\frac{1}{2}}_{2,2})^{-1} T_{2,2}(g^{\frac{1}{2}} I_2 + \lambda A^{\frac{1}{2}}_{2,2})^{-1}\right\|$$
$$+ \|(g^{\frac{1}{2}} I_2 - \lambda A^{\frac{1}{2}}_{2,2})^{-1} T_{2,3} C_{v,0}(\lambda) T_{3,2}(g^{\frac{1}{2}} I_2 + \lambda A^{\frac{1}{2}}_{2,2})^{-1}\| \le \|(g^{\frac{1}{2}} I_2 - \lambda A^{\frac{1}{2}}_{2,2})^{-1} T_{2,2}\| \|(g^{\frac{1}{2}} I_2 + \lambda A^{\frac{1}{2}}_{2,2})^{-1}\|$$
$$+ \|(g^{\frac{1}{2}} I_2 - \lambda A^{\frac{1}{2}}_{2,2})^{-1} T_{2,3}\| \|C_{v,0}(\lambda)\| \|T_{3,2}(g^{\frac{1}{2}} I_2 + \lambda A^{\frac{1}{2}}_{2,2})^{-1}\|. \quad (2.21)$$

The following facts are needed to estimate the terms of (2.21).

Estimating the norm of the resolvent of the self-adjoint operator (see (2.20)), we obtain the inequality

$$\|(I_2 \pm \lambda A^{\frac{1}{2}}_{2,2})^{-1}\| \le (\cos\varepsilon)^{-1}, \quad \lambda \to \infty, \quad \lambda \in \Lambda^{\pm}_{R,\varepsilon}. \quad (2.22)$$

Let $Y \in \mathfrak{S}_\infty$. Then we have

$$\|(I_1 \pm \lambda A_{2,2})^{-1} Y\| = o(1) \quad (2.23)$$

as $\lambda \to \infty,\ \lambda \in \Lambda^{\pm}_{R,\varepsilon}$ (see, e.g., [14]).

It follows from (2.22), (2.23), the compactness of the operators $T_{2,2}$, $T_{2,3}$, and $T_{3,2}$, and the asymptotic relation $\|C_{v,0}(\lambda)\| = O(|\lambda|^{-2})$ $(\lambda \to \infty)$ that

$$\left\|(I_2 - \lambda g^{-\frac{1}{2}} A^{\frac{1}{2}}_{2,2})^{-1}\frac{1}{g}R_{2,2}(\lambda)(I_2 + \lambda g^{-\frac{1}{2}} A^{\frac{1}{2}}_{2,2})^{-1}\right\| = o(1) \quad (\lambda \to \infty,\ \lambda \in \Lambda^{\pm}_{R,\varepsilon}).$$

Thus, the assertion of the theorem follows from Lemma 2.6. □

Lemma 2.7. *Let* $\Lambda_{R,\varepsilon} = \{\,\lambda \mid |\lambda| > R,\ |\arg\lambda| < \varepsilon\,\}$ $(-\pi < \arg\lambda \le \pi)$. *For any (small) positive* ε, *there exists* $R = R(\varepsilon) > N_{0,2}$ *such that the operator-function*

$$D(\lambda) := gI_2 + \lambda^2 A_{2,2} + R_{2,2}(\lambda) \quad (2.24)$$

is continuously invertible in the domain $\Lambda_{R,\varepsilon}$.

The estimate

$$\|D^{-1}(\lambda)\| \le (1+\delta)(\cos\varepsilon)^{-2}, \quad \lambda \to \infty, \quad \lambda \in \Lambda_{R,\varepsilon}, \quad (2.25)$$

is valid for any positive δ.

Proof. Represent the operator-function $D(\lambda)$ as follows:

$$\begin{aligned} D(\lambda) &= (g^{\frac{1}{2}} I_2 - i\lambda A^{\frac{1}{2}}_{2,2})(I_2 + G_{2,0}(\lambda))(g^{\frac{1}{2}} I_2 + i\lambda A^{\frac{1}{2}}_{2,2}),\\ G_{2,0}(\lambda) &:= (g^{\frac{1}{2}} I_2 - i\lambda A^{\frac{1}{2}}_{2,2})^{-1} R_{2,2}(\lambda)(g^{\frac{1}{2}} I_2 + i\lambda A^{\frac{1}{2}}_{2,2})^{-1}. \end{aligned} \quad (2.26)$$

It is obvious that the first and the last factor of $D(\lambda)$ are continuously invertible for $\lambda \in \Lambda_{R,\varepsilon}$. To prove the lemma, it suffices to prove that

$$\|G_{2,0}(\lambda)\| \to 0, \quad \lambda \to \infty, \quad \lambda \in \Lambda_{R,\varepsilon}. \quad (2.27)$$

Using the definition of $R_{2,2}(\lambda)$ given after (2.14), estimate the norm of the operator-function $G_{2,0}(\lambda)$ for $\lambda \to \infty,\ \lambda \in \Lambda_{R,\varepsilon}$:

$$\|(g^{\frac{1}{2}} I_2 - i\lambda A^{\frac{1}{2}}_{2,2})^{-1} R_{2,2}(\lambda)(g^{\frac{1}{2}} I_2 + i\lambda A^{\frac{1}{2}}_{2,2})^{-1}\| \le \|(g^{\frac{1}{2}} I_2 - i\lambda A^{\frac{1}{2}}_{2,2})^{-1} T_{2,2}(g^{\frac{1}{2}} I_2 + i\lambda A^{\frac{1}{2}}_{2,2})^{-1}\|$$
$$+ \|(g^{\frac{1}{2}} I_2 - i\lambda A^{\frac{1}{2}}_{2,2})^{-1} T_{2,3} C_{v,0}(\lambda) T_{3,2}(g^{\frac{1}{2}} I_2 + i\lambda A^{\frac{1}{2}}_{2,2})^{-1}\| \le \|(g^{\frac{1}{2}} I_2 - i\lambda A^{\frac{1}{2}}_{2,2})^{-1} T_{2,2}\| \|(g^{\frac{1}{2}} I_2 + i\lambda A^{\frac{1}{2}}_{2,2})^{-1}\|$$

$$+ \|(g^{\frac{1}{2}} I_2 - i\lambda A_{2,2}^{\frac{1}{2}})^{-1} T_{2,3}\| \|C_{v,0}(\lambda)\| \|T_{3,2}(g^{\frac{1}{2}} I_2 + i\lambda A_{2,2}^{\frac{1}{2}})^{-1}\|. \quad (2.28)$$

Using the representation for norms of resolvents of self-adjoint operators (see (2.20)), we obtain the estimate

$$\|(g^{\frac{1}{2}} I_2 \pm i\lambda A_{2,2}^{\frac{1}{2}})^{-1}\| \leq (\cos\varepsilon)^{-1}, \quad \lambda \to \infty, \quad \lambda \in \Lambda_{R,\varepsilon}. \quad (2.29)$$

Represent the operator-function $C_{v,0}(\lambda)$ outside a disk of finite radius by the Neumann series: $\|C_{v,0}(\lambda)\| = O(|\lambda|^{-2})$, $\lambda \to \infty$. Then use (2.29) and estimates for $\lambda \to \infty$, $\lambda \in \Lambda_{R,\varepsilon}$ (see [14]). This yields $\|(I_1 \pm i\lambda A_{2,2})^{-1} Y\| = o(1)$, $\quad Y \in \mathfrak{S}_\infty$.

Estimate (2.28) implies (2.27). Thus, the operator-function $D^{-1}(\lambda)$ is continuously invertible in the domain $\Lambda_{R,\varepsilon}$. From (2.27) it follows that for any positive δ, there exists a positive $R = R(\delta)$ such that the estimate $\|(I_2 + G_{2,0}(\lambda))^{-1}\| \leq (1 + \delta)$ holds in the domain $\Lambda_{R,\varepsilon}$. □

In general, operators arising in problems of mathematical physics have a power-like asymptotic behavior. In the sequel, it is natural to accept the following conjecture.

Conjecture. *Eigenvalues of the operator* $T := A_{1,1} - A_{1,2} A_{2,2}^{-1} A_{2,1}$ *have a power-like asymptotic behavior.*

Theorem 2.3. *Suppose that the conjecture above is valid and* $\varepsilon > 0$. *Then there exists a sufficiently large* $R = R(\varepsilon)$ *such that spectral problem* (2.11) *has a branch of eigenvalues* $\{\lambda_k^\infty\}_{k=1}^\infty$ *located in the sector* $\Lambda_{R,\varepsilon}$ *and asymptotically behaving as follows*:

$$\lambda_k^\infty = \mu \lambda_k(T^{-1})(1 + o(1)) \quad (k \to \infty). \quad (2.30)$$

Proof. Assume that $R = R(\varepsilon)$ is selected according to the conditions of Lemma 2.7. Then from the second equation of (2.14) it follows that

$$\vec{\varphi} = D^{-1}(\lambda) Q \vec{\psi}_1 - D^{-1}(\lambda) R_{1,2}(\lambda) \vec{\psi}_1. \quad (2.31)$$

Substitute (2.31) in the first equation of (2.14) and divide the obtained expression by $-\lambda\mu$. We obtain the following spectral problem:

$$l_1(\lambda)\vec{\psi}_1 := \left(I_1 - \frac{\lambda}{\mu} T + G_2(\lambda)\right) \vec{\psi}_1 = 0, \quad \vec{\psi}_1 \in \vec{J}_{0,S_1}(\Omega_1, \rho_1), \quad (2.32)$$

$$G_2(\lambda) := -\frac{1}{\lambda\mu}\Big(g B_Q + R_{1,1}(\lambda) + (-g Q^* + R_{1,1}(\lambda))\, D^{-1}(\lambda)\, (Q - R_{2,1}(\lambda))\Big).$$

Obviously, $\|G_2(\lambda)\| \to 0$ $(\lambda \to \infty,\ \lambda \in \Lambda_{R,\varepsilon})$ here. The operator T is self-adjoint, positive, and compact. According to the conjecture, its eigenvalues have a power-like asymptotic behavior. This and the Avakyan theorem (see [1]) imply the proof of the theorem. □

Thus, studying the problem on normal oscillation of a hydraulic system consisting of layers of a viscous and ideal stratified fluids, we see that internal floatability waves (see Lemma 2.3), surface waves (see Theorem 2.2), and internal dissipative waves (see Theorem 2.3) arise in such hydraulic systems. Thus, the spectrum of a composite partially dissipative system is the union of spectra of a conservative subsystem (an ideal fluid) and a dissipative subsystem (a viscous fluid).

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