On Abstract Green's Identity for sesquilinear forms and applications

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$$\{F;(\cdot,\cdot)_F\},\ \{E;(\cdot,\cdot)_E\},\ F\hookrightarrow E \iff \overline{F}=E,\ \|u\|_E\leqslant a\|u\|_F,\quad \forall u\in F.$$

$A:\ \mathcal{D}(A)\subset F\to E$ — the operator of Hilbert pair $(F;E)$

$$(u,Av)_E=(u,v)_F,\quad \forall u\in\mathcal{D}(A)\subset F,\quad v\in F.$$

$$\alpha\in\mathbb{R}\quad\Longrightarrow\quad E^{\alpha}:=\mathcal{D}(A^{\alpha}),$$

$$E=E^0,\quad F=E^{1/2},\quad F^*=E^{-1/2},$$

$$l_v(u):=\langle u,v\rangle_E,\quad u\in F,\quad v\in F^*,\quad |\langle u,v\rangle_E|\leqslant\|u\|_F\cdot\|v\|_{F^*}.$$

$$A:\ F\to F^*,\quad (u,v)_F=\langle u,Av\rangle_E,\quad \forall u,v\in F.$$

Theorem 1.1

$\{E, (\cdot, \cdot)_E\}$, $\{F, (\cdot, \cdot)_F\}$, $\{G, (\cdot, \cdot)_G\}$.

$$\left.\begin{array}{ll} 1^\circ & F \hookrightarrow E, \quad \|u\|_E \leqslant a\|u\|_F, \quad \forall u \in F; \\ 2^\circ & \text{trace operator} \quad \gamma : F \to G_+ \hookrightarrow G, \\ & \|\gamma u\|_G \leqslant b\|u\|_F, \qquad \forall u \in F; \\ 3^\circ & \ker\,\gamma =: N, \quad \overline{N} = E. \end{array}\right\} \Longrightarrow$$

$\exists$ abstract differential expression $Lu \in F^*$ and abstract normal derivative $\partial u \in (G_+)^*$:

$$(\eta, u)_F = \langle \eta, Lu\rangle_E + \langle \gamma\eta, \partial u\rangle_G, \quad \forall \eta, u \in F.$$

Here ∂u is defined uniquelly by $u \in F$ and $Lu \in F^*$.

Ideas of proof.

a) $F = N \oplus M$, $N = \ker\,\gamma$, $\quad M \underset{\gamma_M^{-1}}{\overset{\gamma_M}{\longleftrightarrow}} G_+$ — isometry:

$(\varphi, \psi)_{G_+} := (u, v)_F, \quad \gamma_M u = \varphi, \quad \gamma_M v = \psi, \quad u, v \in M, \quad \gamma_M := \gamma|_M$.
$P_N : F \to N, \quad P_M : F \to M, \quad P_N$ and P_M are orthoprojections.

b) $(\eta_N, u)_F = \langle \eta_N, P_N^* A u\rangle_E =: \langle \eta_N, L_N u\rangle_E, \quad \eta_N = P_N \eta, \ \ u \in F.$

$$\Longrightarrow \left\{ \begin{array}{c} (\eta_N, u_N)_F = \langle \eta_N, L_N u_N\rangle_E, \quad \forall \eta_N, u_N \in N, \\ (\eta_N, u_M)_F = 0 = \langle \eta_N, L_N u_M\rangle_E \quad \Longrightarrow \quad L_N u_M = 0, \ \ u_M = P_M u \in M. \end{array} \right.$$

c) Lu is an extension of $L_N u$ on the whole $F = N \oplus M$,

$$Lu = L_N u + L_M u, \quad L_N : F \to N^*, \quad L_M : F \to M^* := AM,$$

$Lu_M = 0$ (applications), $\quad \forall u_M \in M \ \Longrightarrow \ L_M u_M = 0.$

$$\Psi_u(\eta) := (\eta, u)_F - \langle \eta, L_N u\rangle_E = \Psi_u(\eta_M) = \langle \eta, L_M u\rangle_E + \langle \gamma\eta, \partial u\rangle_G, \quad \eta, u \in F.$$

$\forall\ L_M u \in M^* = AM, \quad \partial u \in (G_+)^*.$

$$0 = \langle \eta_M, L_M u_N\rangle_E + \langle \gamma_M \eta_M, \partial_N u_N\rangle_G, \quad \forall \eta_M \in M, \quad \forall u_N \in N.$$

$Lu := L_N u + L_M u$ is an abstract differential expression;
$\partial u := \partial_M u_M + \partial_N u_N$ is an abstract normal derivative. □

Remark 1.1

$$\left.\begin{array}{c} L(\alpha)u := L_N u + \alpha L_M u = L_N u_N + \alpha L_M u_N, \\ \partial(\alpha)u := \partial_M u_M + \alpha\,\partial_N u_N, \end{array}\right| \quad \Longrightarrow$$

$$(\eta, u)_F = \langle \eta, L(\alpha)u \rangle_E + \langle \gamma\eta, \partial(\alpha)u \rangle_G, \quad \forall \eta, u \in F.$$

Remark 1.2

In applications Lu is defined uniquelly from the physical sence of the investigated problem.

Theorem 1.2 (The second Green's Identity)

a) E, F and G are real spaces:

$$\langle \eta, Lu \rangle_E - \langle u, L\eta \rangle_E = \langle \gamma u, \partial\eta \rangle_G - \langle \gamma\eta, \partial u \rangle_G, \quad \forall \eta, u \in F;$$

b) E, F and G are complex spaces:

$$\langle \eta, Lu \rangle_E - \overline{\langle u, L\eta \rangle}_E = \overline{\langle \gamma u, \partial\eta \rangle}_G - \langle \gamma\eta, \partial u \rangle_G, \quad \forall \eta, u \in F.$$

1.3. The main example:

$\Omega \subset \mathbb{R}^m$, $\Gamma := \partial\Omega$ is Lipschitzian, $E = L_2(\Omega)$, $F = H^1(\Omega) \hookrightarrow\hookrightarrow L_2(\Omega)$ (compactly embedded), $G = L_2(\Gamma)$, $\gamma u := u|_\Gamma$, $\forall u \in H^1(\Omega)$.

Gagliardo theorem

$\gamma : \; H^1(\Omega) \to H^{1/2}(\Gamma) \hookrightarrow\hookrightarrow L_2(\Gamma)$,

$$\|\varphi\|^2_{H^{1/2}(\Gamma)} := \int_\Gamma |\varphi|^2 \, d\Gamma + \int_{\Gamma_x}\int_{\Gamma_y} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{m+1}} \, d\Gamma_x \, d\Gamma_y,$$

a) $\|\gamma u\|_{H^{1/2}(\Gamma)} \leqslant c_1 \|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega)$;

b) $\varphi \in H^{1/2}(\Gamma) \implies u \in H^1(\Omega) : \; \gamma u = \varphi, \quad \|u\|_{H^1(\Omega)} \leqslant c_2 \|\varphi\|_{H^{1/2}(\Gamma)}$.

$N = \ker \gamma = H^1_0(\Omega), \quad M =: H^1_h(\Omega) = \{u \in H^1(\Omega) : u - \Delta u = 0\}$.

Remark 1.3

$(H^{1/2}(\Gamma))^* = H^{-1/2}(\Gamma)$, Γ is Lipschitzian,

$$H^{1/2}(\Gamma) \hookrightarrow\hookrightarrow L_2(\Gamma) \hookrightarrow\hookrightarrow H^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))^*.$$

Remark 1.4

$$H_0^1(\Omega) \hookrightarrow\hookrightarrow L_2(\Omega) \hookrightarrow\hookrightarrow (H_0^1(\Omega))^* \hookrightarrow\hookrightarrow (H^1(\Omega))^*.$$

Here $L_N u := P_N^*(u - \Delta u) \in N^* = (H_0^1(\Omega))^*, \quad P_N^* = AP_N;$

$$(\eta_N, u_N)_{H^1(\Omega)} = \langle \eta_N, L_N u_N \rangle_{L_2(\Omega)}, \;\; \eta_N = P_N \eta, \;\; u_N = P_N u \in H_0^1(\Omega);$$

$$(\eta_N, u_M)_{H^1(\Omega)} = 0 = \langle \eta_N, L_N u_M \rangle_{L_2(\Omega)}, \;\; \eta_N \in H_0^1(\Omega), \;\; u_M \in H_h^1(\Omega).$$

$$Lu := L_N u + L_M u, \quad \forall L_M u \in (H_h^1(\Omega))^*, \quad L_M : \; H^1(\Omega) \to (H_h^1(\Omega))^*,$$

$$L_M u_M = 0, \quad \forall u_M = P_M u \in H_h^1(\Omega).$$

$$(\eta_M, u_M)_{H^1(\Omega)} = \langle \gamma_M \eta_M, \partial_M u_M \rangle_{L_2(\Gamma)},$$

$$\partial_M u_M =: \frac{\partial u_M}{\partial n}\bigg|_\Gamma \in H^{-1/2}(\Gamma).$$

$$0 = \langle \eta_M, L_M u_N \rangle_{L_2(\Omega)} + \langle \gamma_M \eta_M, \partial_N u_N \rangle_{L_2(\Gamma)},$$

$$\partial_N u_N := (\partial u)|_N \in (G_+)^* = H^{-1/2}(\Gamma), \;\; L_M u_N \in M^*.$$

The special case:

$$L_M u_N := P_M^*(u_N - \Delta u_N), \;\; u_N \in H_0^1(\Omega) \quad \Longrightarrow$$
$$Lu := L_N u + L_M u = (P_N^* + P_M^*)(u - \Delta u) = u - \Delta u \in (H^1(\Omega))^*.$$

Theorem 1.3

$L_2(\Omega)$, $H^1(\Omega)$, $L_2(\Gamma)$, $\Gamma = \partial\Omega$, $\gamma u := u|_\Gamma \quad \Longrightarrow$

$$(\eta, u)_{H^1(\Omega)} = \langle \eta, u - \Delta u \rangle_{L_2(\Omega)} + \langle \gamma\eta, \frac{\partial u}{\partial n}\Big|_\Gamma \rangle_{L_2(\Gamma)}, \quad \forall \eta, u \in H^1(\Omega),$$

$$u - \Delta u \in (H^1(\Omega))^*, \quad \frac{\partial u}{\partial n}\Big|_\Gamma \in H^{-1/2}(\Gamma).$$

Remark 1.5

$$L(\alpha)u := P_N^*(u - \Delta u) + \alpha P_M^*(u - \Delta u) \in (H^1(\Omega))^*,$$
$$\partial(\alpha)u := \Big(\frac{\partial u_M}{\partial n}\Big)_\Gamma + \alpha\Big(\frac{\partial u_N}{\partial n}\Big)_\Gamma \in H^{-1/2}(\Gamma),$$

α — is an arbitrary constant.

1°. Uniformly elliptic differential expression:

$$Lu := -\sum_{j,k=1}^{m} \frac{\partial}{\partial x_j}\Big(a_{jk}(x)\frac{\partial u}{\partial x_k}\Big) + a_0(x)u,$$

$$\frac{\partial u}{\partial \nu} := \sum_{j,k=1}^{m} a_{jk}(x)\frac{\partial u}{\partial x_k} n_j, \quad \vec{n} = \sum_{j=1}^{m} n_j \vec{e}_j,$$

$$\|u\|^2_{H^1_{eq}(\Omega)} := \int_\Omega \Big[\sum_{j,k=1}^{m} a_{jk}(x)\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_j} + a_0(x)|u|^2\Big]\, d\Omega \quad \Longrightarrow$$

$$\langle \eta, Lu\rangle_{L_2(\Omega)} = (\eta, u)_{H^1_{eq}(\Omega)} - \langle \gamma\eta, \frac{\partial u}{\partial \nu}\rangle_{L_2(\Gamma)},$$

$$Lu \in (H^1_{eq}(\Omega))^*, \;\; \gamma\eta \in H^{1/2}(\Gamma), \;\; \frac{\partial u}{\partial \nu} \in H^{-1/2}(\Gamma).$$

2°. Generalized Green's Identity for the system of elliptic equations:

$$u(x) := (u_1(x); \ldots ; u_n(x))^{\tau}, \quad \partial_j := \partial/\partial x_j, \quad j = \overline{1, m},$$

$$L_a u := - \sum_{j,k=1}^{m} \partial_j \Big[a_{jk}(x) \partial_k u(x) \Big] + a_0(x) u(x),$$

$$\partial_{\nu_a} u(x) := \sum_{j,k=1}^{m} n_j(x) a_{jk}(x) \partial_k u(x),$$

$$\langle \eta, L_a u \rangle_{L_2(\Omega)} = (\eta, u)_{H^1_a(\Omega)} - \langle \gamma\eta, \partial_{\nu_a} u \rangle_{L_2(\Gamma)}, \quad \forall \eta, u \in H^1_a(\Omega),$$

$$L_a u \in (H^1_a(\Omega))^*, \quad \partial_{\nu_a} u \in (H^{1/2}(\Gamma))^* = H^{-1/2}(\Gamma),$$

$$\|u\|^2_{L_2(\Omega))} := \sum_{r=1}^{n} \|u_r\|^2_{L_2(\Omega)}, \quad \|\varphi\|^2_{L_2(\Gamma))} := \sum_{r=1}^{n} \|\varphi_r\|^2_{L_2(\Gamma)},$$

$$\|u\|^2_{H^1_a(\Omega)} := \int_{\Omega} E(u, u)\, d\Omega + \int_{\Omega} (a_0(x) u) \cdot u \, d\Omega, \quad E(u, u) := \sum a^{rs}_{jk} \partial_j u^r \partial_k u^s.$$

3°. Generalized Green's Identity in the theory of elasticity:

$\Omega \subset \mathbb{R}^3$, $\vec{u}(x)$ is a displacement field of an elastic medium,

$$L\vec{u} := \vec{u} - [\mu\Delta\vec{u} + (\lambda+\mu)\nabla \mathrm{div}\vec{u}], \quad \lambda, \mu > 0.$$

$$\|\vec{u}\|^2_{\vec{H}^1_{eq}(\Omega)} := \mu E(\vec{u}, \vec{u}) + \lambda \int_\Omega |\mathrm{div}\vec{u}|^2 \, d\Omega + \int_\Omega |\vec{u}|^2 \, d\Omega.$$

$$E(\vec{u}, \vec{u}) := \frac{1}{2} \int_\Omega \sum_{j,k=1}^{3} |\tau_{jk}(\vec{u})|^2 \, d\Omega, \quad \tau_{jk}(\vec{u}) := \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j},$$

$$P\vec{u} := \sum_{j,k=1}^{3} (\mu\,\tau_{jk}(\vec{u}) + \lambda\,\mathrm{div}\vec{u}\,\delta_{jk}) \cos(\vec{n}, \widehat{\ } \vec{e}_j)\vec{e}_j,$$

$$\vec{u} = \sum_{j=1}^{3} u_j \vec{e}_j, \quad \gamma\vec{\eta} := \sum_{j=1}^{3} (\gamma\eta_j)\vec{e}_j =: \vec{\eta}|_\Gamma \quad \Longrightarrow$$

$$\langle \vec{\eta}, L\vec{u} \rangle_{\vec{L}_2(\Omega)} = (\vec{\eta}, \vec{u})_{\vec{H}^1_{eq}(\Omega)} - \langle \gamma\vec{\eta}, P\vec{u} \rangle_{\vec{L}_2(\Gamma)}, \quad \forall \vec{\eta}, \vec{u} \in \vec{H}^1_{eq}(\Omega) = \vec{H}^1(\Omega),$$

$$L\vec{u} \in (\vec{H}^1_{eq}(\Omega))^*, \quad \gamma\vec{\eta} \in \vec{H}^{1/2}(\Gamma), \quad P\vec{u} \in (\vec{H}^{1/2}(\Gamma))^* = \vec{H}^{-1/2}(\Gamma).$$

$$\langle \eta, Lu\rangle_E = (\eta, u)_F - \sum_{k=1}^{q} \langle \gamma_k \eta, \partial_k u\rangle_{G_k},$$

$$G = \bigoplus_{k=1}^{q} G_k, \quad \exists (G_+)_k, \quad (G_+)_k^* : (G_+)_k \hookrightarrow G_k \hookrightarrow (G_+)_k^*, \quad k = \overline{1, q}.$$

Let $p_k : G_+ \to \widehat{(G_+)}_k := p_k G_+, \quad p_k^2 = p_k$ is bounded,
$\widehat{\gamma}_k := p_k \gamma, \quad \widehat{\partial}_k := p_k^* \partial, \quad p_k^* : \widehat{(G_+)}_k \to (G_+)^*.$

Theorem 2.1

$$\langle \eta, Lu\rangle_E = (\eta, u)_F - \sum_{k=1}^{q} \langle \widehat{\gamma}_k \eta, \widehat{\partial}_k u\rangle_G, \quad \forall \eta, u \in F.$$

Example.

$\Omega \subset \mathbb{R}^m, \quad \Gamma = \partial\Omega$ is Lipschitzian, $\Gamma = \Gamma_1 \cup \Gamma_2, \quad \operatorname{dist}(\Gamma_1, \Gamma_2) =: d > 0.$

$$\varphi \in H^{1/2}(\Gamma) \Longrightarrow \varphi = \begin{cases} \varphi_1 & (\text{on } \Gamma_1), \\ \varphi_2 & (\text{on } \Gamma_2), \end{cases}$$

$$p_1\varphi := \begin{cases} \varphi_1 & (\text{on } \Gamma_1), \\ 0 & (\text{on } \Gamma_2), \end{cases} \quad p_2\varphi := \begin{cases} 0 & (\text{on } \Gamma_1), \\ \varphi_2 & (\text{on } \Gamma_2). \end{cases}$$

$$p_1^2 = p_1, \quad \widehat{H}^{1/2}(\Gamma_1) := \{\varphi = (\varphi_1; 0) : \ \varphi_1 \in H^{1/2}(\Gamma_1)\} \subset H^{1/2}(\Gamma).$$

Lemma 2.1

$$p_k \in \mathcal{L}(H^{1/2}(\Gamma); \widehat{H}^{1/2}(\Gamma_k)), \quad k = 1, 2.$$

Remark 2.1

$$\langle \widehat{\gamma}_1 \eta, \widehat{\partial}_1 u \rangle_G = \int\limits_\Gamma \eta \frac{\partial u}{\partial n} \, d\Gamma, \quad \eta = 0 \ \ (\text{on} \ \ \Gamma_2).$$

$$\int\limits_{\Gamma_1} (\eta|_{\Gamma_1}) \Big(\frac{\partial u}{\partial n}\Big)_{\Gamma_1} d\Gamma_1 \qquad - \quad ? \qquad \longmapsto \qquad \langle \eta|_{\Gamma_1}, \frac{\partial u}{\partial n}\Big|_{\Gamma_1} \rangle_{L_2(\Gamma_1)}.$$

For mathphys. problems we have

$$p_k = \omega_k \rho_k, \quad k = \overline{1, q},$$

$\rho_k : G_+ \to (G_+)_k$ — abstract restriction operator;
$\omega_k : \ (G_+)_k \to \widehat{(G_+)}_k$ — abstract extension operator by zero finction;
$\rho_k \omega_k = I_k \quad (\text{on} \ \ (G_+)_k), \quad k = \overline{1, q},$
ρ_k and ω_k are bounded.

For above example:

$$\rho_k \varphi := \varphi|_{\Gamma_k}, \quad \forall \varphi \in H^{1/2}(\Gamma),$$

$$\omega_1 \varphi_1 := \begin{cases} \varphi_1 & \text{(on } \Gamma_1) \\ 0 & \text{(on } \Gamma_2) \end{cases}, \quad \omega_2 \varphi_2 := \begin{cases} 0 & \text{(on } \Gamma_1) \\ \varphi_2 & \text{(on } \Gamma_2) \end{cases}.$$

Theorem 2.2

$$\langle \eta, Lu \rangle_E = (\eta, u)_F - \sum_{k=1}^{q} \langle \gamma_k \eta, \partial_k u \rangle_{G_k}, \quad \forall \eta, u \in F.$$

$$\gamma_k \eta := \rho_k \gamma \eta \in (G_+)_k, \quad \partial_k u := \omega_k^* \partial u \in (G_+)_k^*.$$

For above example:

Lemma 2.2

$$\|\rho_k\|_{H^{1/2}(\Gamma) \to H^{1/2}(\Gamma_k)} \leqslant 1.$$

Remark 2.2

$$\|\psi_k\|_{H^s(\Gamma_k)} := \inf \{ \|\widehat{\psi}\|_{H^s(\Gamma)} : \ \widehat{\psi}|_{\Gamma_k} = \psi_k \}, \quad |s| \leqslant 1.$$

For extension operators: $\exists$ Rychkov operator, namely:

Lemma 2.3 (Rychkov [10], see also Agranovich [9], $\partial \Gamma_k$ are Lipschitzian.)

$\exists \ \mathcal{E} : \ H^s(\Gamma_k) \to H^s(\Gamma),$

$$\|\mathcal{E}\psi\|_{H^s(\Gamma)} \leqslant c \|\psi\|_{H^s(\Gamma_k)}, \ \forall \psi \in H^s(\Gamma_k), \ |s| \leqslant 1.$$

Consider the following auxiliary mixed boundary value problems:

$$w - \Delta w = 0 \ \text{(in } \Omega), \quad w = 0 \ \text{(on } \Gamma \setminus \Gamma_k), \quad \frac{\partial w}{\partial n}\Big|_{\Gamma_k} = \psi_k \ \text{(on } \Gamma_k).$$

Lemma 2.4

$\exists !$ weak solution to this problem if and only if $\psi_k \in H^{-1/2}(\Gamma_k)$, and then

$$w =: \widetilde{T}_k \psi \in H^1_{0,\Gamma\setminus\Gamma_k}(\Omega) \cap H^1_h(\Omega) =: H^1_{h,\Gamma_k}(\Omega),$$

$$H^1_{0,\Gamma\setminus\Gamma_k}(\Omega) := \{u \in H^1(\Omega) : u = 0 \ \text{(on } \ \Gamma \setminus \Gamma_k)\}.$$

Introduce

$$\widetilde{H}^{1/2}(\Gamma_k) := \{\widetilde{\varphi}_k = \gamma_k w : w \in H^1_{h,\Gamma_k}(\Omega)\} \subset H^{1/2}(\Gamma_k).$$

Lemma 2.5

$\widetilde{H}^{1/2}(\Gamma_k)$ is dense in $L_2(\Gamma_k)$.

As a corollary:

Lemma 2.6

The extension operator ω_k with $\mathcal{D}(\omega_k) = \widetilde{H}^{1/2}(\Gamma_k)$ and $\mathcal{R}(\omega_k) \subset H^{1/2}(\Gamma)$ is bounded and

$$\|\omega_k \widetilde{\varphi}_k\|_{H^{1/2}(\Gamma_k)} \leqslant c_1 \|\widetilde{\varphi}_k\|_{\widetilde{H}^{1/2}(\Gamma_k)}, \quad \forall \widetilde{\varphi}_k \in \widetilde{H}^{1/2}(\Gamma_k),$$

c_1 — from Gagliardo theorem.

Definition

$$\widehat{H}^1(\Omega) := H_0^1(\Omega) \oplus \{(\dot{+})_{k=1}^q H^1_{h,\Gamma_k}(\Omega)\} \subset H^1(\Omega),$$

$$\widehat{H}^{1/2}(\Gamma) := \{\varphi \in H^{1/2}(\Gamma) : \rho_k \varphi = \varphi|_{\Gamma_k} \in \widetilde{H}^{1/2}(\Gamma_k), \ \ k = \overline{1,q}\} \subset H^{1/2}(\Gamma).$$

Theorem 2.3

$$\langle \eta, u - \Delta u \rangle_{L_2(\Omega)} = (\eta, u)_{H^1(\Omega)} - \sum_{k=1}^{q} \langle \gamma_k \eta, \partial_k u \rangle_{L_2(\Gamma_k)}, \quad \forall \eta, u \in \widehat{H}^1(\Omega),$$

$$u - \Delta u \in (\widehat{H}^1(\Omega))^*, \gamma_k \eta := \eta|_{\Gamma_k} \in \widetilde{H}^{1/2}(\Gamma_k), \ \ \partial_k u := \frac{\partial u}{\partial n}\Big|_{\Gamma_k} \in H^{-1/2}(\Gamma_k), \ \ k = \overline{1,q}.$$

2.2. The final formulation of Abstract Green's Identity for mixed boundary value problem.

Assumptions: $(G_+)_k \hookrightarrow G_k \hookrightarrow (G_+)^*_k, \quad k = \overline{1,q}, \qquad G = \bigoplus_{k=1}^{q} G_k,$

$p_k = \omega_k \rho_k, \quad k = \overline{1,q}, \quad \rho_k\omega_k = (I_+)_k, \quad \sum_{k=1}^q p_k = I_+,$
or $p^*_k = \rho^*_k \omega^*_k$, $\ldots$, all operators are bounded
(ω_k is an extension operator, but not by zero function!).

Theorem 2.4

$\langle \eta, Lu\rangle_E = (\eta, u)_F - \sum_{k=1}^q \langle \gamma_k\eta, \partial_k u\rangle_{G_k}, \quad \forall \eta, u \in F,$
$\gamma_k\eta := \rho_k\gamma\eta \in (G_+)_k, \quad \partial_k u := \omega^*_k \partial u \in (G_+)^*_k.$

Example: $E = L_2(\Omega)$, $F = H^1(\Omega)$, $G = L_2(\Gamma)$, $\Gamma = \partial\Omega$ — is Lipschitzian,
$L_2(\Gamma) = \oplus_{k=1}^q L_2(\Gamma_k)$, $\partial\Gamma_k$ is Lipschitzian.

Theorem 2.5

$(\eta, u)_{H^1(\Omega)} = \langle \eta, u - \Delta u\rangle_{L_2(\Omega)} + \sum_{k=1}^q \langle \gamma_k\eta, \partial_k u\rangle_{L_2(\Gamma_k)}, \quad \forall \eta, u \in H^1(\Omega),$

$$\gamma_k\eta := \eta|_{\Gamma_k} \in H^{1/2}(\Gamma_k), \quad \partial_k u := \frac{\partial u}{\partial n}\Big|_{\Gamma_k} \in H^{-1/2}(\Gamma_k), k = \overline{1,q}, \quad u - \Delta u \in (H^1(\Omega))^*.$$

Theorem 2.6

The spaces $\widehat{H}^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ are isomorphic; the spaces $\widehat{H}^1(\Omega)$ and $H^1(\Omega)$ are isomorphic, too.

3. Abstract Green's Identity for sesquilinear forms.
3.1. Sesquilinear bounded forms.

$\Phi(\eta, u) : F \times F \to \mathbb{C}$, Φ — is linear by η and semilinear by u,

$$|\Phi(\eta, u)| \leqslant c_1 \|\eta\|_F \cdot \|u\|_F, \ \ \forall \eta, u \in F, \ \ c_1 > 0.$$

$$F \hookrightarrow E \hookrightarrow F^*,$$

$\Phi(\eta, u) \longleftrightarrow A : F \to F^*$, $\quad \Phi(\eta, u) = \langle \eta, Au \rangle_E, \ \ \forall \eta, u \in F,$

$$\|A\|_{F \to F^*} \leqslant c_1$$

.

$$\Phi^*(\eta, u) := \overline{\Phi(u, \eta)} \text{— adjoint form,}$$

$$\Phi(\eta, u) = \Phi^*(\eta, u) \text{— self-adjoint form.}$$

$$\operatorname{Re}\Phi(u,u) = \operatorname{Re}\langle u, Au\rangle_E \geqslant c_2\|u\|_F^2, \quad c_2 > 0, \quad \forall u \in F.$$

Lemma 3.1

$\exists!$ representation:

$$\Phi(\eta,u) = (Q_F\eta,u)_F = (\eta, Q_F^* u)_F, \quad \forall \eta, u \in F,$$

$$Q_F \in \mathcal{L}(F), \quad Q_F^{-1} \in \mathcal{L}(F).$$

Lemma 3.2 (Lax, Milgram)

$$\Phi(\eta,u) \longleftrightarrow A, \quad \exists\, A^{-1} \in \mathcal{L}(F^*, F).$$

3.3. On representation of a nonsymmetric uniformly accretive form.

$\Phi(\eta, u) \neq \Phi^*(\eta, u) \quad \Longrightarrow$

$$\Phi_R(\eta, u) := \frac{1}{2}\Big[\Phi(\eta, u) + \Phi^*(\eta, u)\Big] = \Phi_R^*(\eta, u),$$

$$\Phi_I(\eta, u) := \frac{1}{2i}\Big[\Phi(\eta, u) - \Phi^*(\eta, u)\Big] = \Phi_I^*(\eta, u),$$

$$\Phi(\eta, u) = \Phi_R(\eta, u) + i\Phi_I(\eta, u).$$

$$c_2\|u\|_F^2 \leqslant \Phi_R(u, u) =: \|u\|_{F_0}^2 \leqslant c_1\|u\|_F^2, \quad \forall u \in F.$$

$(F_0;\ E)$ — Hilbert pair of spaces, $\quad A_0 \longleftrightarrow (F_0;\ E)$,

$$E = E^0, \quad F_0 = E^{1/2} = \mathcal{D}(A_0), \quad F_0^* = E^{-1/2} = \mathcal{R}(A_0),$$

$$A_0^{1/2} \in \mathcal{L}(F_0; E), \quad A_0^{1/2} \in \mathcal{L}(E; F_0^*),$$

$$(\eta, u)_{F_0} = (A_0^{1/2}\eta, A_0^{1/2}u)_E = \langle \eta, A_0 u\rangle_E, \quad \forall \eta, u \in F_0.$$

$$\Phi_I(\eta, u) = (QA_0^{1/2}\eta, A_0^{1/2}u)_E = (A_0^{1/2}\eta, QA_0^{1/2}u)_E, \quad Q = Q^* \in \mathcal{L}(E).$$

Lemma 3.3

$$\Phi(\eta, u) = \langle \eta, Au\rangle_E,$$

$$A = A_0^{1/2}(I - iQ)A_0^{1/2} \ \in \mathcal{L}(F_0, F_0^*),$$

$$A^{-1} = A_0^{-1/2}(I - iQ)^{-1}A_0^{-1/2} \ \in \mathcal{L}(F_0^*, F_0), \quad A^* = A_0^{1/2}(I + iQ)A_0^{1/2}.$$

3.4. On Abstract Green's Identity for sesquilinear forms

$$(\eta, u)_{F_0} := \Phi_R(\eta, u) \quad \Longrightarrow$$

$\{E, F_0, G, \gamma\} \quad \Longrightarrow$ Abstract Green's Identity (symmetric case):

$$\Phi_R(\eta, u) = \langle \eta, L_0 u \rangle_E + \langle \gamma\eta, \partial_0 u \rangle_G, \ \ \forall \eta, u \in F_0 = F,$$

$$F_0 = N_0 \oplus M_0, \ \ N_0 = \ker \gamma, \ \ L_0 u := L_{0,N_0} u + L_{0,M_0} u,$$

$$\partial_0 u = \partial_{M_0} u_{M_0} + \partial_{N_0} u_{N_0}, \quad u = u_{N_0} + u_{M_0} \in F_0, \quad \partial_0 u \in (G_+)^*.$$

$$\Phi(\eta, u) = \langle \eta, L u \rangle_E + \langle \gamma\eta, \partial u \rangle_G, \ \ \forall \eta, u \in F_0. \qquad (?)$$

$Q \to 0 \quad \Longrightarrow \quad$ transition to symmetric case.

$$F_0 = N_0 \oplus M_0 = N(\dot{+})M, \ \ N = \ker \gamma, \ \ M \neq M_0.$$

$P_N, \ \ P_M = I_{F_0} - P_N \quad$ are projections on N and M.

Tne main condition:

$$\Phi(\eta_N, u_M) = 0, \quad \forall \eta_N = P_N \eta \in N, \quad \forall u_M = P_M u \in M.$$

$$Lu = L_N u_N + L_M u_N = L u_N,$$

$$L_M u_M = 0, \quad L_N u_M = 0, \quad \forall u_M \in M.$$

$$\exists! \quad P_M u = P_{M_0} u + i(I_{N_0} - iP_{N_0} Q_0 P_{N_0})^{-1} (P_{N_0} Q_0 P_{M_0})(P_{M_0} u), \quad \forall u \in F_0,$$

$$P_N u = I_{F_0} - P_M u, \quad P_{N_0} \text{ and } P_{M_0} \text{ are orthoprojections onto } N_0 \text{ and } M_0,$$

$$Q_0 := A_0^{-1/2} Q A_0^{1/2} = Q_0^* \in \mathcal{L}(F_0), \quad P_M^2 = P_M \in \mathcal{L}(F_0), \quad P_N^2 = P_N \in \mathcal{L}(F_0),$$

$\partial u := \partial_M u_M + \partial_N u_N$:

$$\Phi(\eta_M, u_M) = \langle \eta_M, L_M u_M \rangle_E + \langle \gamma_M \eta_M, \partial_M u_M \rangle_G,$$
$$\Phi(\eta_N, u_N) = \langle \eta_N, L_N u_N \rangle_E, \quad L_N : F_0 \to N^*,$$
$$\Phi(\eta_M, u_N) = \langle \eta_M, L_M u_N \rangle_E + \langle \gamma_M \eta_M, \partial_N u_N \rangle_G, \quad L_M : F_0 \to M^*,$$
$$\Phi(\eta_N, u_M) = 0 \quad \Longrightarrow$$

Theorem 3.1 (the First Abstract Green's Identity for sesquilinear forms)

$\exists \; Lu = L_N u + L_M u = L u_N \in F_0^*$ — abstract differential expression,
$\partial u = \partial_N u_N + \partial_M u_M \in (G_+)^*$:
$\Phi(\eta, u) = \langle \eta, Lu \rangle_E + \langle \gamma \eta, \partial u \rangle_G, \quad \forall \eta, u \in F_0 = F;$
conormal derivative ∂u is defined by $u \in F$ and $Lu \in F^*$ uniquelly.

Remark 3.1

The same result for $\Phi^*(\eta, u)$, i.e., $\Phi^*(\eta, u) = \langle \eta, L_* u \rangle_E + \langle \gamma \eta, \partial_* u \rangle_G, \quad \forall \eta, u \in F_0 = F,$
$P_{M_*} u = P_{M_0} u - i(I_{N_0} + iP_{N_0} Q_0 P_{N_0})^{-1} (P_{N_0} Q_0 P_{M_0})(P_{M_0} u), \quad \forall u \in F_0,$
$P_{N_*} u = I_{F_0} - P_{M_*}, \quad F_0 = N_0 \oplus M_0 = N_* (\dot{+}) M_*, \quad N_* = N_0, \quad M_* \neq M_0.$

Some applications: abstract boundary value and spectral problems (according to Green's Identity)

1°. *Nonhomogeneous Newmann problem for Poisson equation*:

$Lu = f, \quad \partial u = \psi \quad \Longrightarrow$

$f \in F_0^*, \quad \psi \in (G_+)^* \;\Longrightarrow \quad \exists!\, u = A^{-1} f + T_M \psi,$

$A = A_0^{1/2}(I - iQ)A_0^{1/2}, \quad T_M : (G_+)^* \to M \subset F_0 \iff Lw = 0, \quad \partial u = \psi.$

2°. *Dirichlet problem for Poisson equation*:

$Lu = f, \quad \gamma u = \varphi,$

$f \in N^*, \quad \varphi \in G_+ \;\Longrightarrow \quad \exists!\, u = A_{00}^{-1} f + \gamma_M^{-1}\varphi, \quad A_{00} = P_N^* A P_N : N \to N^*.$

3°. *The third boundary value problem (Newton – Newmann problem)*:

$Lu = f, \quad \partial u + \alpha\gamma u = \psi, \quad \alpha : G_+ \to (G_+)^*, \quad \langle \varphi, \alpha\varphi \rangle_G \geqslant 0.$

(The same result.)

4°. *Abstract mixed boundary value problem: on the base of Abstract Green's Identity of the form*

$$\Phi(\eta, u) = \langle \eta, Lu \rangle_E + \sum_{k=1}^{q} \langle \gamma_k \eta, \partial_k u \rangle_{G_k}, \quad \eta, u \in F_0.$$

Spectral problems:

$Lu = \lambda u, \quad \gamma u = 0$ (Dirichlet problem);

$Lu = \lambda u, \quad \partial u = 0$ (Newmann problem);

$Lu = \lambda u, \quad \partial u + \alpha\gamma u = 0$ (Newton problem);

$Lw = 0, \quad \partial w + \alpha\gamma w = \lambda\gamma w$ (Steklov problem);

$Lu = \lambda u, \quad \partial u + \alpha\gamma u = \lambda\gamma u$; (Stefan problem);

$Lu + \lambda u = 0, \quad \partial u + \alpha\gamma u = \mu\gamma u, \quad \lambda, \mu \in \mathbb{C}$ (M. Agranovich problem);

$Lu = \lambda u, \quad \lambda(\partial u + \alpha\gamma u) = \gamma u$ (S. Krein problem);

$Lu + \lambda^2 u = 0, \quad \partial u + \alpha\gamma u = \lambda\gamma u$ (Chueshov problem).

Transmission problems (M. Agranovich, etc.), the same approach for corresponding Abstract Green's Identity.

Thank you for your attention.