On Abstract Green's Identity for sesquilinear forms

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1°. Let for arbitrary Hilbert spaces E, F and G (with introduced scalar products) the following assumptions be fulfilled.

i) The space F is boundedly embedded in E, $F \hookrightarrow E$.

ii) There exists an abstract trace operator $\gamma : F \to G$ and $\mathcal{R}(\gamma) =: G_+ \hookrightarrow G$.

iii) There exists a sesquilinear form $\Phi(\eta, u)$, $\eta, u \in F$, such that $|\Phi(\eta, u)| \leqslant c_1\|\eta\|_F\|u\|_F$, $\operatorname{Re}\Phi(u,u) \geqslant c_2\|u\|_F^2$, $c_1 \geqslant c_2 > 0$.

Theorem 1. If assumptions i) – iii) hold then the following Abstract Green's Identity is valid:

$$\Phi(\eta, u) = \langle \eta, Lu\rangle_E + \langle \gamma\eta, \partial u\rangle_G, \quad \forall \eta, u \in F, \quad Lu \in F^*, \quad \partial u \in (G_+)^*. \tag{1}$$

Here Lu is an abstract differential expression corresponding to the form $\Phi(\eta, u)$ and ∂u is an abstract conormal derivative. They are defined uniquely by the data of the problem. □

2°. Let for projections p_k, $k = \overline{1,q}$, acting in G_+, the following assumptions be fulfilled.

iv) $p_k = \omega_k\rho_k$ (or $p_k^* = \rho_k^*\omega_k^*$) where ρ_k is a bounded restriction operator acting from G_+ on $(G_+)_k := \rho_k G_+$, and ω_k is bounded extension operator acting from $(G_+)_k$ onto $p_k G_k$, $k = \overline{1,q}$.

v) $\rho_k\omega_k = (I_+)_k$ (an identity operator on $(G_+)_k$), $k = \overline{1,q}$.

vi) $\sum\limits_{k=1}^{q} p_k = I_+$ (an identity operator on G_+).

Theorem 2. Under assumptions iv) – vi) Abstract Green's Identity (1) has the form

$$\begin{gathered}\Phi(\eta, u) = \langle \eta, Lu\rangle_E + \sum_{k=1}^{q}\langle \gamma_k\eta, \partial_k u\rangle_{G_k}, \quad \forall \eta, u \in F, \\ \gamma_k\eta := \rho_k\gamma\eta \in (G_+)_k, \quad \partial_k u := \omega_k^*\partial u \in (G_+)_k^*, \quad (G_+)_k \hookrightarrow G_k \hookrightarrow (G_+)_k^*, \quad k = \overline{1,q}.\end{gathered} \tag{2}$$

Here $\gamma_k = \rho_k\gamma$ is an abstract trace operator on the k-th part of the boundary and $\partial_k = \omega_k^*\partial$ is a corresponding conormal derivative. □

3°. We consider some examples of applications of formulas (1) and (2) for classical boundary value problems and for problems in hydrodynamics and elasticity theory.