Complete Second Order Volterra Integro-Differential Equations in Hilbert Space

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$$A\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + Bu + \sum_{k=1}^{m}\int_0^t G_k(t,s)A_k u(s)ds = f(t),$$

$$u(0) = u^0, \quad u'(0) = u^1,$$

$u = u(t)$ with values in $\mathcal{H}$,

$F \gg 0$, $G = G^*$, $B \gg 0$, A_k, $k = \overline{1,m}$, are unbounded operators,

$G_k(t,s)$ is an operator function of variables t,s with values in $\mathcal{L}(\mathcal{H})$.

Let F, G, B and A_k be comparable.

1°. The strongly damped dynamic systems: $\mathcal{D}(F) \subset \mathcal{D}(B)$, $\mathcal{D}(A_k) \supset \mathcal{D}(F)$.

2°. The weakly damped dynamic systems: $\mathcal{D}(F) \subset \mathcal{D}(B^{1/2})$, $\mathcal{D}(A_k) \supset \mathcal{D}(B^{1/2})$.

3°. The average damped dynamic systems: $\mathcal{D}(B) \subset \mathcal{D}(F) \subset \mathcal{D}(B^{1/2})$,
$\mathcal{D}(A_k) \supset \mathcal{D}(B^{1/2})$.

0.2. The Cauchy Problem for First Order Diff. Equation

$\mathcal{E}$ is a Banach space.

$$\frac{du}{dt} = Au + f(t), \quad u(0) = u^0, \quad t \geqslant 0. \tag{1}$$

$u = u(t)$ — an unknown function with values in $\mathcal{E}$, $f(t)$ — a given function, $u^0 \in \mathcal{E}$, $A \in \mathcal{L}(\mathcal{E})$, $\mathcal{D}(A) \subset \mathcal{E}$ — a gener. of strong contin. or analyt. semigroup $\mathcal{U}(t)$, $t \in \mathbb{R}_+$.

Definition of s.s.

A function $u(t)$ with values in $\mathcal{E}$ is called a s.s. to problem (1) on $[0, T]$ **if**:
a) $u(t) \in \mathcal{D}(A)$, $t \in [0, T]$, b) $Au(t) \in C([0, T]; \mathcal{E})$; c) $u(t) \in C^1([0, T]; \mathcal{E})$;
d) $\forall\, t \in [0, T]$ the eq. (3) holds true; e) initial cond. (1) is satisfied.

Introduce $W_p^1([0, T]; \mathcal{E})$: $\|u(t)\|_{W_p^1([0,T];\mathcal{E})} := \sum_{k=0}^{1} \Big(\int_0^T \|u^{(k)}(t)\|^p dt \Big)^{1/p}, \; p > 1.$

Theorem (S.Yakubov)

Let in problem (1) the following cond. be satisfied:
a) A is a gener. of C_0-semigroup $\mathcal{U}(t)$, b) $u^0 \in \mathcal{D}(A)$, c) $f(t) \in W_p^1([0, T]; \mathcal{E})$, $p > 1$.
Then (1) has uniq. s.s. $u(t)$ on $[0, T]$, and it represents by formula

$$u(t) = \mathcal{U}(t)u^0 + \int_0^t \mathcal{U}(t - s) f(s) ds. \tag{2}$$

$\mathcal{E}$ is a Banach space.

$$\frac{du}{dt} = A_0 u + \sum_{k=1}^{m} \int_0^t G_k(t,s) A_k u(s) ds + f(t), \quad u(0) = u^0. \tag{3}$$

A_0 — a gener. of strong contin. or holomorphic semogroup, $u(t)$ — an unknown function with values in $\mathcal{E}$, $f(t)$ — a given function, bounded oper.-functions $G_k(t,s)$ and oper. A_k, $k = 1, 2, \ldots m$, act in $\mathcal{E}$.

Definition of s.s.

A function $u(t)$ with values in $\mathcal{E}$ is called a s.s. to (3) on $[0,T]$ **if**:

a) $u(t) \in C^1([0,T];\mathcal{E}) \cap C([0,T];\mathcal{D}(A_0))$;
b) all items in (3) are contin. ($\in C([0,T];\mathcal{E})$);
c) $\forall\, t \in [0,T]$ the eq. (3) holds true;
d) $u(0) = u^0$.

0.4. The theorem on strong solvability

Theorem 0.1

Let in problem (3) the following conditions are satisfied:

$$\mathcal{D}(A_k) \supset \mathcal{D}(A_0), \quad k = 1, 2, \dots m, \tag{4}$$

$$G_k(t,s), \frac{\partial G_k}{\partial t}(t,s) \in C(\Delta_T; \mathcal{L}(\mathcal{E})), \ \Delta_T := \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\}. \tag{5}$$

$$u^0 \in \mathcal{D}(A_0). \tag{6}$$

Let A_0 is a gener. of C_0-semigroup, and

$$f(t) \in C^1([0,T]; \mathcal{E}). \tag{7}$$

Then problem (3) has uniq. s.s. on $[0,T]$.

Theorem 0.2

Let in problem (3) the assumpt. of Th. 0.1 and instead (7) the cond. $f(t) \in W_p^1([0,T]; \mathcal{E})$, $p > 1$, be satisfied. Then problem (3) has iniq. s.s. on $[0,T]$.

Theorem 0.3

Let in problem (3) cond. (4), (5), (6) be satisfied, A_0 — a gener. of analyt. semigroup, and $f(t) \in C^\alpha([0,T]; \mathcal{E})$, $0 < \alpha \leqslant 1$. Then problem (3) has iniq. s.s. on $[0,T]$.

$$A\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + Bu + \sum_{k=1}^{m}\int_0^t G_k(t,s)A_k u(s)ds = f(t),$$

$$u(0) = u^0, \quad u'(0) = u^1, \tag{8}$$

$$\frac{1}{2}\Big(A\frac{du}{dt}, \frac{du}{dt}\Big) \quad \Longrightarrow \quad A = A^* > 0.$$

$$\frac{1}{2}(Bu, u) \quad \Longrightarrow \quad B = B^* \geqslant 0; \quad B \geqslant \gamma_B I,\ \gamma_B \in \mathbb{R}.$$

$$\frac{1}{2}\Big(F\frac{du}{dt}, \frac{du}{dt}\Big) \quad \Longrightarrow \quad F = F^* \geqslant \gamma_F I,\ \gamma_F \in \mathbb{R}.$$

$$iG\frac{du}{dt}: \quad G \in \mathcal{L}(\mathcal{H}). \quad (G \notin \mathcal{L}(\mathcal{H}))$$

Assumptions:

$A = I$:

$$\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + Bu + \sum_{k=1}^{m}\int_0^t G_k(t,s)A_k u(s)ds = f(t),$$

"*Shortened*" problem:

$$\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + Bu = f(t), \quad u(0) = u^0, \quad u'(0) = u^1, \tag{11}$$

Assumptions:

$$F = F^* \gg 0, \quad B = B^* \gg 0, \quad G \in \mathcal{L}(\mathcal{H}). \tag{12}$$

Def. 1.1

$u = u(t) \in \mathcal{H}$ is a strong solution to (11) $\quad t \in [0, T] \quad$ if:
a) $u(t) \in \mathcal{D}(B) \quad \forall t \in [0, T]$ and $Bu(t) \in C([0, T]; \mathcal{H})$;
b) $\dfrac{du}{dt} \in \mathcal{D}(F)$ and $F\dfrac{du}{dt} \in C([0, T]; \mathcal{H})$;
c) $u(t) \in C^2([0, T]; \mathcal{H})$;
d) eq. (11) is satisfied $\quad t \in [0, T]$;
e) init. cond. (11) are satisfied.

$\Longrightarrow$ **Necessary conditions of existence s.s.:**

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(F), \quad f(t) \in C([0, T]; \mathcal{H}).$$

Transition to First Order Differential Equations System

Substitution: $v(t) \in C^2([0,T];\mathcal{H})$

$$\frac{dv}{dt} = -iB^{1/2}u(t), \quad v(0) = 0, \tag{13}$$

$\mathcal{D}(B^{1/2}) \supset \mathcal{D}(B) \supset \mathcal{D}(F)$, $F\dfrac{du}{dt} \in C([0,T];\mathcal{H}) \implies B^{1/2}u(t) \in C^1([0,T];\mathcal{H})$,

$B^{1/2}\dfrac{du}{dt} \in C([0,T];\mathcal{H}) \implies v(t) \in C^2([0,T];\mathcal{H}) \implies$

$$\frac{d^2v}{dt^2} = -iB^{1/2}\frac{du}{dt}, \quad v'(0) = -iB^{1/2}u^0. \tag{14}$$

$$\frac{d^2}{dt^2}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F+iG & iB^{1/2} \\ iB^{1/2} & 0 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}, \tag{15}$$

$$u(0) = u^0, \quad u'(0) = u^1, \quad v(0) = 0, \quad v'(0) = -iB^{1/2}u^0. \tag{16}$$

Substitution: $y(t) = (u'(t); v'(t))^\tau \in \mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$

$$\frac{dy}{dt} + \mathcal{A}_0 y = f_0(t), \quad y(0) = y^0, \tag{17}$$

$$y^0 := (u^1; -iB^{1/2}u^0)^\tau, \quad f_0(t) := (f(t); 0)^\tau, \tag{18}$$

$$\mathcal{A}_0 := \begin{pmatrix} F+iG & iB^{1/2} \\ iB^{1/2} & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_0) := \mathcal{D}(F) \oplus \mathcal{D}(B^{1/2}). \tag{19}$$

Lemma 1.1

$\mathcal{A}_0$ is accretive operator in $\mathcal{H}^2$: $\quad Re(\mathcal{A}_0 y, y)_{\mathcal{H}^2} \geqslant 0, \quad y \in \mathcal{D}(\mathcal{A}_0).$

Ideas of proof: $G = G^*$, $B^{1/2} = (B^{1/2})^*$, $F \gg 0 \quad \Longrightarrow Re(\mathcal{A}_0 y, y)_{\mathcal{H}^2} = (F y_1, y_1) \geqslant 0.$ $\square$

Substitution: $\quad y(t) = \mathrm{e}^{at} z(t), \quad a > 0,$

$$\frac{dz}{dt} + \mathcal{A}_a z = f_a(t), \quad z(0) = y^0, \tag{20}$$

$$\mathcal{A}_a := \mathcal{A}_0 + aI = \begin{pmatrix} F_a + iG & iB^{1/2} \\ iB^{1/2} & aI \end{pmatrix}, \; F_a := F + aI, \; f_a(t) := \mathrm{e}^{-at}(f(t); 0)^\tau. \tag{21}$$

$$\mathrm{Re}(\mathcal{A}_a z, z)_{\mathcal{H}^2} \geqslant a\|z\|^2_{\mathcal{H}^2}, \quad z \in \mathcal{D}(\mathcal{A}_a) = \mathcal{D}(\mathcal{A}_0). \tag{22}$$

$$Q_a := B^{1/2} F_a^{-1/2}, \quad Q_a^+ = F_a^{-1/2} B^{1/2}, \quad \mathcal{D}(Q_a^+) := \mathcal{D}(B^{1/2}).$$

Lemma 1.2

$Q_a \in \mathcal{L}(\mathcal{H})$, $Q_a^+ = Q_a^* | \mathcal{D}(B^{1/2})$, $\quad \overline{Q_a^+} = Q_a^* \in \mathcal{L}(\mathcal{H})$.

Ideas of proof: $F \gg 0$, $B \gg 0$, $\mathcal{D}(F) \subset \mathcal{D}(B) \implies \mathcal{D}(F^\alpha) \subset \mathcal{D}(B^\alpha)$, $0 < \alpha < 1 \implies$ $\mathcal{D}(B^{1/2}) \supset \mathcal{D}(F^{1/2}) = \mathcal{D}(F_a^{1/2}) \implies \mathcal{D}(Q_a) = \mathcal{D}(B^{1/2} F_a^{-1/2}) = \mathcal{H} \implies Q_a \in \mathcal{L}(\mathcal{H})$.
$u \in \mathcal{H}$, $v \in \mathcal{D}(B^{1/2})$: $\quad (Q_a u, v) = (B^{1/2} F_a^{-1/2} u, v) = (u, F_a^{-1/2} B^{1/2} v) = (u, Q_a^+ v)$
$\implies Q_a^+ \subset Q_a^*$.
$Q_a \in \mathcal{L}(\mathcal{H}) \implies Q_a^* \in \mathcal{L}(\mathcal{H}) \implies Q_a^+ \in \mathcal{L}(\mathcal{D}(B^{1/2})) \implies Q_a^+ = Q_a^* | \mathcal{D}(B^{1/2})$.
$\mathcal{D}(B^{1/2})$ dense in $\mathcal{H} \implies \overline{Q_a^+} = Q_a^*$.

Th. 1.1

$\mathcal{A}_a$: $\mathcal{D}(\mathcal{A}_a) = \mathcal{D}(F) \oplus \mathcal{D}(B^{1/2})$

1°. factorization in Sh.-Fr. form

$$\mathcal{A}_a = \begin{pmatrix} I & 0 \\ iQ_aF_a^{-1/2} & I \end{pmatrix} \begin{pmatrix} F_a & 0 \\ 0 & aI + Q_aQ_a^+ \end{pmatrix} \begin{pmatrix} I & iF_a^{-1/2}Q_a^+ \\ 0 & I \end{pmatrix} + \begin{pmatrix} iG & 0 \\ 0 & 0 \end{pmatrix};$$

2°. factorization in symm. multipl. form

$$\mathcal{A}_a = \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I + iF_a^{-1/2}GF_a^{-1/2} & iQ_a^+ \\ iQ_a & aI \end{pmatrix} \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix}. \tag{23}$$

$\mathcal{A} := \overline{\mathcal{A}_a}$ is maximal uniformly accretive operator: $Re(\mathcal{A}z, z)_{\mathcal{H}^2} \geqslant a\|z\|^2_{\mathcal{H}^2}, \quad z \in \mathcal{D}(\mathcal{A})$.

1°. factorization in Sh.-Fr. form

$$\mathcal{A} = \begin{pmatrix} I & 0 \\ iQ_aF_a^{-1/2} & I \end{pmatrix} \begin{pmatrix} F_a & 0 \\ 0 & aI + Q_aQ_a^* \end{pmatrix} \begin{pmatrix} I & iF_a^{-1/2}Q_a^* \\ 0 & I \end{pmatrix} + \begin{pmatrix} iG & 0 \\ 0 & 0 \end{pmatrix};$$

2°. factorization in symm. multipl. form

$$\mathcal{A} = \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I + iF_a^{-1/2}GF_a^{-1/2} & iQ_a^* \\ iQ_a & aI \end{pmatrix} \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix}. \tag{24}$$

$\mathcal{A}$: $\mathcal{D}(\mathcal{A}) := \left\{ z = (z_1; z_2)^\tau \in \mathcal{H}^2 : z_1 + iF_a^{-1/2}Q_a^*z_2 \in \mathcal{D}(F_a) \right\}$:

$$\mathcal{A}z = \begin{pmatrix} F_a(z_1 + iF_a^{-1/2}Q_a^*z_2) + iGz_1 \\ iQ_aF_a^{1/2}z_1 + az_2 \end{pmatrix} = \begin{pmatrix} F_a^{1/2}(F_a^{1/2}z_1 + iQ_a^*z_2) + iGz_1 \\ iQ_aF_a^{1/2}z_1 + az_2 \end{pmatrix}.$$

Remark

$$Re\left(\left(\begin{array}{cc} I+iF_a^{-1/2}GF_a^{-1/2} & iQ_a^* \\ iQ_a & aI \end{array}\right)\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right),\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)\right) =$$

$$= \|z_1\|^2 + a\|z_2\|^2 \geqslant \min(1;a)\|z\|^2_{\mathcal{H}^2}, \quad \forall z \in \mathcal{H}^2,$$

Corrolary 1.1

$(-\mathcal{A})$ is a generator of a semigroup $\mathcal{U}(t)$:

$$\|\mathcal{U}(t)\| \leqslant e^{-at}, \quad t \geqslant 0. \tag{25}$$

Ideas of proof: $\mathcal{A}_0$ is accretive, $\mathcal{A} - a\mathcal{I} = \overline{\mathcal{A}_a} - a\mathcal{I}$ is maximal accretive $\Longrightarrow \quad -\mathcal{A} + a\mathcal{I}$ is maximal dissipative operator and a generator of semigroup $\mathcal{V}(t) \Longrightarrow \mathcal{U}(t) := \mathcal{V}(t)\mathrm{e}^{-at}$ has a generator $(-\mathcal{A})$.

□

Th. 1.2

Let in (10), (11), (12) :

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(F), \tag{26}$$

and one of the conditions is satisfied

$$f(t) \in W_p^1([0,T];\mathcal{H}), \quad p > 1, \tag{27}$$

$$F^{-1} \in \mathfrak{S}_\infty(\mathcal{H}), \quad f(t) \in C^\alpha([0,T];\mathcal{H}), \quad 0 \leqslant \alpha < 1. \tag{28}$$

Then:

1°. Problem (11), (10), (12) has a unique strong solution in $[0,T]$.
2°. Problem (20)–(21) has a unique strong solution in $[0,T]$.
3°. Cauchy problem

$$\frac{dz}{dt} + \mathcal{A}z = f_a(t), \quad z(0) = y^0 = (u^1; -iB^{1/2}u^0)^\tau, \tag{29}$$

has a unique strong solution in $[0,T]$.

Ideas of proof:

1°. If (11), (10), (12) has unique s.s. on $[0,T]$ $\Longrightarrow$ (15)–(16) and (20)–(21) have unique s.s. on $[0,T]$, and

$$z(0) = y^0 = (u^1; -iB^{1/2}u^0)^\tau \in \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A}_a), \tag{30}$$

and $\widehat{f}(t) = \mathrm{e}^{-at}(f(t);0)^\tau$ satisfies (27) or (28) in $\mathcal{H}^2$.

If (20)–(21) has unique s.s. $z(t) \Longrightarrow$ for u^0, u^1 and $f(t)$ (26)–(28)) be satisfied.
Return to (15)–(16), (11), (10), (12) $\Longrightarrow$ (11), (10), (12) has s.s. on $[0,T]$ $\Longrightarrow$
it is sufficient to prove: if (26)–(28) are satisfied, then (20)–(21) and (29) both have s.s. on $[0,T]$.

2°. Let (26), (27) $\Longrightarrow$ $f_a(t) \in W_p^1([0,T];\mathcal{H}^2)$, $p>1$, and (30).
$\mathcal{A}$ — max. uniformly accr. (Th. 1.1) $\Longrightarrow$ (29) has unique s.s. on $[0,T]$. $\Longrightarrow$ $\forall\, t \in [0,T]$

$$\frac{dz_1}{dt} + F_a(z_1(t) + iF_a^{-1/2}Q_a^* z_2(t)) + iGz_1(t) = \mathrm{e}^{-at}f(t), \tag{31}$$

$$\frac{dz_2}{dt} + az_2(t) + iQ_aF_a^{1/2}z_1(t) = 0, \tag{32}$$

$$z_1(0) = u^1, \quad z_2(0) = -iB^{1/2}u^0, \tag{33}$$

and all items in (31), (32) $\in C([0,T];\mathcal{H})$,

$$z_1(t) + iF_a^{-1/2}Q_a^* z_2(t) \in \mathcal{D}(F_a) = \mathcal{D}(F), \quad \forall\, t \in [0,T]. \tag{34}$$

(32), (33) $\implies$ $z_2(t) = -i\mathrm{e}^{-at}B^{1/2}u^0 - i\int_0^t \mathrm{e}^{-a(t-s)}Q_aF_a^{1/2}z_1(s)ds, \quad t \in [0,T].$

$z_2(t) \mapsto$ (31) $\implies$ $z_1(t)$ is s.s. of Cauchy Problem

$$\frac{dz_1}{dt} + F_a\Big(z_1(t) + \mathrm{e}^{-at}F_a^{-1/2}Q_a^*B^{1/2}u^0 + \int_0^t \mathrm{e}^{-a(t-s)}F_a^{-1/2}Q_a^*Q_aF_a^{1/2}z_1(s)ds\Big) + \\ + iGz_1(t) = \mathrm{e}^{-at}f(t), \quad z_1(0) = u^1.$$

$\implies$ $z_1(t) \in C^1([0,T];\mathcal{H})$ and $\forall\, t \in [0,T]$

$$\varphi(t) := z_1(t) + \int_0^t \mathrm{e}^{-a(t-s)}F_a^{-1/2}Q_a^*Q_aF_a^{1/2}z_1(s)ds + \varphi_0(t) \quad \in C([0,T];\mathcal{D}(F_a)), \quad (35)$$

$$\varphi_0(t) := \mathrm{e}^{-at}F_a^{-1/2}Q_a^*B^{1/2}u^0 \quad \in C([0,T];\mathcal{D}(F_a)). \quad (36)$$

Rewrite (35) in the form

$$z_1(t) + \int_0^t \mathrm{e}^{-a(t-s)}F_a^{-1/2}Q_a^*Q_aF_a^{1/2}z_1(s)ds = \varphi_1(t) := \varphi(t) - \varphi_0(t), \quad (37)$$

$\varphi_1(t) \in C([0,T];\mathcal{D}(F_a))$.
(37) is integr. Volterra eq. in $C([0,T];\mathcal{H}(F_a))$,

$$\|u\|_{\mathcal{H}(F_a)} := \|F_au\|_{\mathcal{H}}, \quad \forall u \in \mathcal{D}(F_a).$$

$\mathcal{D}(B) \supset \mathcal{D}(F) = \mathcal{D}(F_a) \implies \forall\, u \in \mathcal{D}(F_a) = \mathcal{H}(F_a)$:

$$Tu := F_a^{-1/2} Q_a^* Q_a^{1/2} F_a^{1/2} u = F_a^{-1/2} Q_a^+ B^{1/2} u = F_a^{-1} B u \in \mathcal{D}(F_a).$$

$\implies$ $T|_{\mathcal{H}(F_a)}$ is bounded in $\mathcal{H}(F_a)$ $\quad (T := F_a^{-1/2} Q_a^* Q_a F_a^{1/2})$.

$\implies$ kernel function of operator (37) is contin. with values in $\mathcal{L}(\mathcal{H}(F_a))$.

$\varphi_1(t) \in C([0,T];\mathcal{H}(F_a)) \implies$ (37) has uniq. s.s. $z_1(t) \in C([0,T];\mathcal{H}(F_a))$.

(31) $\implies$ if (27) is satisfied, then the theorem is proved.

3) Let (28) be satisfied $\implies$ $f_a(t) = \mathrm{e}^{-at}(f(t);0)^\tau \in C^\alpha([0,T];\mathcal{H}^2), \quad 0 < \alpha \leqslant 1.$

Using factorization for $\mathcal{A}$, rewrite (29) in the form

$$\frac{dz}{dt} + (\mathcal{I} + \mathcal{S}_1)\mathcal{A}_{00}(\mathcal{I} + \mathcal{S}_2)z + i\mathcal{G}z = f_a(t), \quad z(0) = y^0, \tag{38}$$

$$\mathcal{S}_1 := \begin{pmatrix} 0 & 0 \\ iQ_a F_a^{-1/2} & 0 \end{pmatrix}, \quad \mathcal{S}_2 := \begin{pmatrix} 0 & iF_a^{-1/2} Q_a^* \\ 0 & 0 \end{pmatrix}, \tag{39}$$

$$\mathcal{A}_{00} := \operatorname{diag}(F_a; aI + Q_a Q_a^*), \quad \mathcal{G} := \operatorname{diag}(G;0). \tag{40}$$

Here:

$F^{-1} \in \mathfrak{S}_\infty(\mathcal{H}) \implies F_a^{-1} \in \mathfrak{S}_\infty(\mathcal{H})$ and $F_a^{-1/2} \in \mathfrak{S}_\infty(\mathcal{H}) \implies \mathcal{S}_i \in \mathfrak{S}_\infty(\mathcal{H}^2)$.

$(\mathcal{I} + \mathcal{S}_i)$ has triangle form $\implies$ $(\mathcal{I} + \mathcal{S}_i)$ have bounded inverse operators

$$(\mathcal{I} + \mathcal{S}_i)^{-1} = \mathcal{I} - \mathcal{S}_i, \quad i = 1,2. \tag{41}$$

Substitution:

$$(\mathcal{I}+\mathcal{S}_2)z(t) =: w(t). \tag{42}$$

Using (41), (38) and $(\mathcal{I}+\mathcal{S}_2)f_a(t) \equiv f_a(t) \implies$ Cauchy Problem

$$\frac{dw}{dt} + (\mathcal{I}+\mathcal{S}_2)(\mathcal{I}+\mathcal{S}_1)\mathcal{A}_{00}w + i(\mathcal{I}+\mathcal{S}_2)\mathcal{G}(\mathcal{I}+\mathcal{S}_2)^{-1}w = f_a(t),$$
$$w(0) = (\mathcal{I}+\mathcal{S}_2)y^0, \tag{43}$$

Here:

$\mathcal{A}_{00} = (\mathcal{A}_{00})^* \gg 0$ is unbounded, $\mathcal{D}(\mathcal{A}_{00}) = \mathcal{D}(F) \oplus \mathcal{H}$.

$\mathcal{S}_i \in \mathfrak{S}_\infty(\mathcal{H}^2) \implies (\mathcal{I}+\mathcal{S}_2)(\mathcal{I}+\mathcal{S}_1) =: (\mathcal{I}+\mathcal{S}), \quad \mathcal{S} \in \mathfrak{S}_\infty(\mathcal{H}^2)$.

$\implies (\mathcal{I}+\mathcal{S}_2)(\mathcal{I}+\mathcal{S}_1)\mathcal{A}_{00} = (\mathcal{I}+\mathcal{S})\mathcal{A}_{00}$ — is a weakly pert. of $\mathcal{A}_{00} = (\mathcal{A}_{00})^* \gg 0$.

$\implies -(\mathcal{I}+\mathcal{S})\mathcal{A}_{00}$ is a gener. of analit. semigroup in a sector $\supset (t \geqslant 0)$ ([15], c. 92, 183)

$i(\mathcal{I}+\mathcal{S}_2)\mathcal{G}(\mathcal{I}+\mathcal{S}_2)^{-1}$ — bounded in $\mathcal{H}^2 \implies$

$$\mathcal{B} := -[(\mathcal{I}+\mathcal{S}_2)(\mathcal{I}+\mathcal{S}_1)\mathcal{A}_{00} + i(\mathcal{I}+\mathcal{S}_2)\mathcal{G}(\mathcal{I}+\mathcal{S}_2)^{-1}] \tag{44}$$

is a generator of analit. semigroup ([18], c. 66), and (43) is an abstract parab. eq.

If (26) $\implies y^0 \in \mathcal{D}(\mathcal{A}) \implies w(0) = (\mathcal{I}+\mathcal{S}_2)y^0 \in \mathcal{D}(\mathcal{A}_{00})$.

If (28) $\implies f_a(t) \in C^\alpha([0,T];\mathcal{H}^2)$, $\alpha > 0 \implies$ (43) has uniq. s.s. $w(t)$ on $[0,T]$.

Using inverse subst. (42), (41) and back from (43) to (38) and (29):

if (26), (28) $\implies$ (29) has uniq. s.s. $z(t)$ on $[0,T]$.

Repeat from (31) ...

Corrolary 1.2

If (10), (12), $f(t) \equiv 0$, then (26) are necess. and suffic. for solvability of homogeneous problem (9).

Corrolary 1.3

The necess. and suffic. cond. of solvability of nonhomogeneous problem (9): requir. on $f(t)$ are the same that in Cauchy Problem for first order diff. eq. with oper. coeff.—gener. of analit. semigroup.

1.3. The case: diss. oper. and pot. en. oper. are bounded from below, gyrosc. oper. is bounded

Assumptions:

$$F \geqslant \gamma_F I, \quad B \geqslant \gamma_B I, \quad \gamma_F, \gamma_B \in \mathbb{R}, \quad G \in \mathcal{L}(\mathcal{H}). \tag{45}$$

Idea: Trans. (11), (10), (45), the main oper. matrice properties are the same of $\mathcal{A}_a$.

$$B = (B + bI) - bI =: B_b - bI, \quad b \in \mathbb{R}, \tag{46}$$

Choosing b:

$$b + \gamma_B =: \alpha_B > 0 \iff B_b \geqslant \alpha_B I. \tag{47}$$

$$\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + B_b u = f_u(t) =: f(t) + bu(t), \qquad u(0) = u^0, \quad u'(0) = u^1, \tag{48}$$

$$\frac{dv}{dt} = -iB_b^{1/2}u(t), \quad v(0) = 0 \qquad \Longrightarrow \frac{d^2v}{dt^2} = -B_b^{1/2}\frac{du}{dt}, \quad v'(0) = -iB_b^{1/2}u^0.$$

The Cauchy Problem for $u(t)$, $v(t)$:

$$\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + iB_b^{1/2}\frac{dv}{dt} = f_u(t) \quad u'(0) = u^1,$$

$$\frac{d^2v}{dt^2} + iB_b^{1/2}\frac{du}{dt} = 0, \quad v'(0) = -iB_b^{1/2}u^0.$$

In matrice form:

$$\frac{dy}{dt} + \mathcal{A}_{0,b} y = f_{0,u}(t), \quad y(0) = y^0, \tag{49}$$

$$y(t) = \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}, \; f_{0,u}(t) = \begin{pmatrix} f_u(t) \\ 0 \end{pmatrix}, \; y^0 = \begin{pmatrix} u^1 \\ -iB_b^{1/2} u^0 \end{pmatrix}, \tag{50}$$

$$\mathcal{A}_{0,b} = \begin{pmatrix} F + iG & iB_b^{1/2} \\ iB_b^{1/2} & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_{0,b}) := \mathcal{D}(F) \oplus \mathcal{D}(B_b^{1/2}).$$

Substitution:

$$y(t) = \mathrm{e}^{at} z(t), \quad a > 0, \tag{51}$$

Choosing a:

$$a + \gamma_F =: \alpha_F > 0 \implies F_a := F + aI \geqslant \alpha_F I. \tag{52}$$

Instead (49):

$$\frac{dz}{dt} + \mathcal{A}_{a,b} z = f_{a,u}(t) := \mathrm{e}^{-at} f_{0,u}(t), \quad z(0) = y^0, \tag{53}$$

$$\mathcal{A}_{a,b} := \mathcal{A}_{0,b} + a\mathcal{I} = \begin{pmatrix} F_a + iG & iB_b^{1/2} \\ iB_b^{1/2} & aI \end{pmatrix}, \tag{54}$$

$$\mathcal{D}(\mathcal{A}_{a,b}) = \mathcal{D}(\mathcal{A}_{0,b}) := \mathcal{D}(F_a) \oplus \mathcal{D}(B_b^{1/2}). \tag{55}$$

Here:

$$\mathrm{Re}(\mathcal{A}_{a,b} z, z)_{\mathcal{H}^2} = (F_a z_1, z_1)_{\mathcal{H}} + a\|z_2\|^2_{\mathcal{H}} \geqslant c\|z\|^2_{\mathcal{H}^2}, \; c := \min(\alpha_F; a) > 0. \tag{56}$$

General properties of $\mathcal{A}_{a,b}$ are the same ones of $\mathcal{A}_a$.

$$Q_{a,b} := B_b^{1/2} F_a^{-1/2}, \quad Q_{a,b}^{+} := F_a^{-1/2} B_b^{1/2}, \quad \mathcal{D}(Q_{a,b}^{+}) = \mathcal{D}(B_b^{1/2}), \tag{57}$$

$$\mathcal{D}(F) = \mathcal{D}(F_a) \subset \mathcal{D}(B_b) = \mathcal{D}(B), \quad \text{Heintz ineq.} \implies \quad \mathcal{D}(F_a^{1/2}) \subset \mathcal{D}(B_b^{1/2}) \implies$$

$$Q_{a,b}^{+} = Q_{a,b}^{*} | \mathcal{D}(B_b^{1/2}), \quad \overline{Q_{a,b}^{+}} = Q_{a,b}^{*} \in \mathcal{L}(\mathcal{H}).$$

$\implies -\mathcal{A} := -\overline{\mathcal{A}_{a,b}}$ — a gener. of a contracting strong continuous semigroup $\mathcal{U}(t)$

$$\|\mathcal{U}(t)\| \leqslant \mathrm{e}^{-ct}, \quad c > 0, \quad t \geqslant 0,$$

c — const from (56).

Theorem 1.3

Let in Cauchy Problem (11), (10) (45) cond. (26), (27) be satisfied:

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(F), \quad f(t) \in W_p^1([0,T];\mathcal{H}), \quad p > 1. \tag{58}$$

Then this problem has a uniq. s.s. on $[0, T]$.

Ideas of proof:

1) Similarly Th. 1.2: consider Cauchy Problem

$$\frac{dz}{dt} + \mathcal{A}z = f_{a,u}(t), \quad z(0) = y^0, \tag{59}$$

with $\mathcal{A} = \overline{\mathcal{A}_{a,b}}$ and

$$f_{a,u}(t) = \mathrm{e}^{-at}(f(t) + bu(t); 0)^\tau, \quad y^0 := (u^1; -iB_b^{1/2}u^0)^\tau. \tag{60}$$

(58) $\Longrightarrow$

$$u^0 \in \mathcal{D}(B_b),\ u^1 \in \mathcal{D}(F_a),\ \mathrm{e}^{-at}(f(t); 0)^\tau \in W_p^1([0,T]; \mathcal{H}^2),\ p > 1, \tag{61}$$

$\Longrightarrow$

$$y^0 \in \mathcal{D}(\mathcal{A}_{a,b}) \subset \mathcal{D}(\mathcal{A}).$$

$\Longrightarrow$ if the cond.

$$f_{a,u}(t) \in W_p^1([0,T]; \mathcal{H}^2),$$

is satisfied, then Cauchy Problem (59) has a uniq. s.s.

$$z(t) = \mathcal{U}(t)y^0 + \int\limits_0^t \mathcal{U}(t-s)f_{a,u}(s)ds, \quad t \in [0,T]. \tag{62}$$

2) Represent $f_{a,u}(t)$ from (60) in the form

$$f_{a,u}(t) = \mathrm{e}^{-at}(f(t);0)^{\tau} + b\mathrm{e}^{-at}(u(t);0)^{\tau} =: f_{a,0}(t) + b\mathrm{e}^{-at}\Big(u^0 + \int_0^t u'(\xi)d\xi;0\Big)^{\tau} =$$

$$= f_{a,0}(t) + b\mathrm{e}^{-at}(u^0;0)^{\tau} + b\mathrm{e}^{-at}P_1\int_0^t \Big(\frac{du}{d\xi}(\xi);\frac{dv}{d\xi}(\xi)\Big)^{\tau}d\xi$$

$$=: f_{a,b}(t) + b\mathrm{e}^{-at}\int_0^t P_1(\mathrm{e}^{a\xi}z(\xi))d\xi, \quad t\in[0,T], \quad (63)$$

here P_1 — orthopr. from $\mathcal{H}^2$ on $\mathcal{H}$.
(63) $\mapsto$ (62) $\Longrightarrow$ an integr. eq. for an unknown function $z(t)$:

$$z(t) = \mathcal{U}(t)y^0 + \int_0^t \mathcal{U}(t-s)f_{a,b}(s)ds + \int_0^t \mathcal{U}(t-s)ds\int_0^s b\mathrm{e}^{-a(s-\xi)}P_1z(\xi))d\xi =$$

$$= \Big\{\mathcal{U}(t)y^0 + \int_0^t \mathcal{U}(t-s)f_{a,b}(s)ds\Big\} + b\int_0^t\Big\{\int_\xi^s \mathrm{e}^{-a(s-\xi)}\mathcal{U}(t-s)P_1ds\Big\}z(\xi)d\xi$$

$$=: \varphi_0(t) + \int_0^t V(t,\xi)z(\xi)d\xi. \quad (64)$$

$$\Longrightarrow \quad \text{Volterra integr. eq.} \quad z(t) - \int_0^t V(t,\xi)z(\xi)d\xi = \varphi_0(t), \quad t \in [0,T], \tag{65}$$

here oper.-funct. $V(t,\xi)$ is a strong contin. by t, ξ, with values in $\mathcal{L}(\mathcal{H}^2)$.

3) in (65) (Th. of S.Yakubov) $\varphi_0(t)$ — s.s. to (59) with $f_{a,u}(t) \mapsto f_{a,0}(t) \quad \Longrightarrow$
$\varphi_0(t) \in C^1([0,T];\mathcal{H}^2) \quad \Longrightarrow \quad$ integr. eq. (65) has a uniq. solution $z(t) \in C([0,T];\mathcal{H}^2)$.
$\varphi_0(t)$ and items in (65) $\in C([0,T];\mathcal{H}^2) \quad \Longrightarrow \quad z(t) \in C^1([0,T];\mathcal{H}^2)$. Actually:

$$\frac{d}{dt}\int_0^t V(t,\xi)z(\xi)d\xi = b\frac{d}{dt}\int_0^t \Big(\int_\xi^t \mathrm{e}^{-as}\mathcal{U}(t-s)P_1 ds\Big)\mathrm{e}^{a\xi}z(\xi)d\xi =$$

$$= |t-s=\eta, \ -ds = d\eta| = b\frac{d}{dt}\Big(\mathrm{e}^{-at}\int_0^t \Big(\int_0^{t-\xi} \mathrm{e}^{a\eta}\mathcal{U}(\eta)P_1 d\eta\Big)\mathrm{e}^{a\xi}z(\xi)d\xi\Big) =$$

$$= b\Big\{-a\mathrm{e}^{-at}\int_0^t \Big(\int_0^{t-\xi} \mathrm{e}^{a\eta}\mathcal{U}(\eta)P_1 d\eta\Big)\mathrm{e}^{a\xi}z(\xi)d\xi + \int_0^t \mathcal{U}(t-\xi)P_1 z(\xi)d\xi\Big\}, \tag{66}$$

$z(t)$ — contin. $\quad \Longrightarrow \quad$ in right side of (66) — contin. by t.
Then, if (58) are satisfied $\quad \Longrightarrow \quad z(t) \in C^1([0,T];\mathcal{H}^2) \quad \Longrightarrow \quad z(t)$ — s.s. of (59).
4) Repeat from (31) ...

Consider the case closed to (28).

Theorem 1.4

Let in Cauchy Problem (9), (10), (45) cond. (26), (27) are satisfied:

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(F), \quad f(t) \in C^{\alpha}([0,T];\mathcal{H}), \quad \alpha > 0,$$

$$(F_a)^{-1} := (F + aI)^{-1} \in \mathfrak{S}_{\infty}(\mathcal{H}).$$

Then this problem has a uniq. s.s. on $[0,T]$.

Ideas of proof: Combine the base steps of Th.1.2 and Th.1.3.

1) In (11), (10), (45) the same trans. as in Th.1.3, and below (45).
Namely, choose a, b from (47), (52) and trans. to (53).
For $\mathcal{A}_{a,b}$ and $\overline{\mathcal{A}_{a,b}} = \mathcal{A}$ Th. 1.1 is true.
In particular, we have

$$\mathcal{A} = (\mathcal{I} + \mathcal{S}_1)\mathcal{A}_{00}(\mathcal{I} + \mathcal{S}_2) + i\mathcal{G}, \tag{67}$$

where $\mathcal{S}_i$, $i = 1, 2$, $\mathcal{A}_{00}$ и $\mathcal{G}$ are defined in (39), (40), $Q_a \mapsto Q_{a,b}$, $Q_a^* \mapsto Q_{a,b}^*$ $\Longrightarrow$

$$\frac{dz}{dt} + (\mathcal{I} + \mathcal{S}_1)\mathcal{A}_{00}(\mathcal{I} + \mathcal{S}_2)z + i\mathcal{G}z = f_{a,u}(t) = \mathrm{e}^{-at}(f(t) + bu(t); 0)^{\tau}, \quad z(0) = y^0, \tag{68}$$

that closed to problem (38).

2) Substitution in (68): $z(t) = (\mathcal{I} + \mathcal{S}_2)^{-1} w(t) \quad \Longrightarrow$ The Cauchy Problem for $w(t)$:

$$\frac{dw}{dt} + (\mathcal{I} + \mathcal{S}_2)(\mathcal{I} + \mathcal{S}_1)\mathcal{A}_{00} w + i(\mathcal{I} + \mathcal{S}_2)\mathcal{G}(\mathcal{I} + \mathcal{S}_2)^{-1} w = f_{a,u}(t) =$$

$$= f_{a,b}(t) + b\mathrm{e}^{-at} \int_0^t \mathrm{e}^{a\xi} P_1 w(\xi) d\xi, \quad w(0) = (\mathcal{I} + \mathcal{S}_2) y^0. \quad (69)$$

Here we have operator — a gener. of some analit. semigroup $\mathcal{U}(t)$ (Th. 1.2).
3) Let $f_{a,u}(t)$ is known $\Longrightarrow$

$$w(t) = \mathcal{U}(t)(\mathcal{I} + \mathcal{S}_2) y^0 + \int_0^t \mathcal{U}(t-s) f_{a,b}(s) ds +$$

$$+ b \int_0^t \mathrm{e}^{-as} \mathcal{U}(t-s) \Big(\int_0^s \mathrm{e}^{a\xi} P_1 w(\xi) d\xi \Big) ds. \quad (70)$$

Here $\varphi_0(t) = \mathcal{U}(t)(\mathcal{I} + \mathcal{S}_2) y^0 + \int_0^t \mathcal{U}(t-s) f_{a,b}(s) ds$ — s.s. of (69) (step 3) Th. 1.2, substitution $f_{a,u}(t) \mapsto f_{a,b}(t) \quad \Longrightarrow \quad \varphi_0(t) \in C^1([0,T]; \mathcal{H}^2)$.
Repeat step 3) Th. 1.2 ... $\Longrightarrow$ $\exists$ uniq. s.s. $w(t) \in C^1([0,T]; \mathcal{H}^2)$ of (70).

4) (69) $\xrightarrow{\text{inverse subst.}}$ (59) $\Longrightarrow$ (59) has a uniq. s.s. $z(t)$ on $[0,T]$.

Repeat step 2) Th. 1.2: $\Longrightarrow$ (59) (with $\mathcal{A}$) $\longrightarrow$ (53) (with $\mathcal{A}_{a,b}$) $\longrightarrow$ (11), (10), (45).

Below: G is unbounded and is subordinated to $F_a^{1/2}$ or to F.
The simple case:

Theorem 1.5

Let in (11) the conditions

$$\mathcal{D}(F) \subset \mathcal{D}(B),\ F \geqslant \gamma_F I,\ B \geqslant \gamma_B I,\ B_b := B + bI \geqslant \alpha_B I,\ \alpha_B > 0,$$

$$F_a := F + aI \geqslant \alpha_F I, \quad \alpha_F > 0, \quad \mathcal{D}(G) \supset \mathcal{D}(F_a^{1/2}), \tag{71}$$
$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(F),$$

be satisfied, and it be satisfied one of the conditions

$$f(t) \in W_p^1([0,T];\mathcal{H}), \quad p > 1, \tag{72}$$

$$f(t) \in C^{\alpha}([0,T];\mathcal{H}),\ 0 < \alpha < 1,\ F_a^{-1} \in \mathfrak{S}_\infty(\mathcal{H}),\ GF_a^{-1/2} \in \mathfrak{S}_\infty(\mathcal{H}). \tag{73}$$

Then Cauchy Problem (11) has a uniq. s.s. on $[0,T]$.

Ideas of proof:
From last cond. (71) $\Longrightarrow$ $\mathcal{D}(GF_a^{-1/2}) = \mathcal{H}$ and $GF_a^{-1/2}$ — bounded $\Longrightarrow$ combin. Th. 1.2 – Th. 1.4 .

1°. Let (72) be satisfied.

Trans. (46) – (55) in problem (11) $\Longrightarrow$ $GF_a^{-1/2} \in \mathcal{L}(\mathcal{H})$ $\Longrightarrow$ (54), (55).

Then for $\mathcal{A} = \overline{\mathcal{A}_{a,b}}$

$$\mathcal{A} = \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I + iF_a^{-1/2} G F_a^{-1/2} & iQ_{a,b}^* \\ iQ_{a,b} & aI \end{pmatrix} \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix}.$$

Here $GF_a^{-1/2}$ is bounded.

For $z = (z_1; z_2)^\tau \in \mathcal{D}(\mathcal{A})$:

$$z_1 \in \mathcal{D}(F_a^{1/2}) \subset \mathcal{D}(G), \quad iF_a^{-1/2} G F_a^{-1/2} F_a^{1/2} z_1 = iF_a^{-1/2} G z_1 \in \mathcal{D}(F_a^{1/2}). \quad (74)$$

$$\Longrightarrow \quad \mathcal{D}(\mathcal{A}) = \Big\{ z = (z_1; z_2)^\tau \in \mathcal{H}^2 : z_1 + iF_a^{-1/2} Q_{a,b}^* z_2 \in \mathcal{D}(F_a) \Big\}.$$

$\Longrightarrow$ Repeat from (59) to end (Th. 1.3) .

2°. Let (73) be satisfied.

Represent. $\mathcal{A} = \overline{\mathcal{A}_{a,b}}$ in Sh.-Fr. form

$$\mathcal{A} = \begin{pmatrix} I & 0 \\ iQ_{a,b}F_a^{-1/2} & I \end{pmatrix} \begin{pmatrix} F_a & 0 \\ 0 & aI + Q_{a,b}Q_{a,b}^* \end{pmatrix} \begin{pmatrix} I & iF_a^{-1/2}Q_{a,b}^* \\ 0 & I \end{pmatrix} + i \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix},$$

$\longrightarrow$ problem (59) $\longrightarrow$

$$\frac{dz}{dt} + (\mathcal{I} + \mathcal{S}_1)\mathcal{A}_{00}(\mathcal{I} + \mathcal{S}_2)z + i\mathcal{G}z = f_{a,u}(t), \ z(0) = y^0 = (u^1; -iB_b^{1/2}u^0)^\tau,$$

here $\mathcal{S}_i$, $\mathcal{G}$ def. in (39), (40) with subst. $Q_a \mapsto Q_{a,b}$, $Q_a^* \mapsto Q_{a,b}^*$.

Trans. from Th. 1.4.

$(\mathcal{I} + \mathcal{S}_2)\mathcal{G}(\mathcal{I} + \mathcal{S}_2)^{-1}$ is compactly subordinated to $-(\mathcal{I} + \mathcal{S}_2)(\mathcal{I} + \mathcal{S}_1)\mathcal{A}_{00}$, because $-(\mathcal{I} + \mathcal{S}_2)(\mathcal{I} + \mathcal{S}_1)\mathcal{A}_{00}$ — a gener. of an analit. semigroup (as in Th. 1.4).
Actually,

$$(\mathcal{I} + \mathcal{S}_2)\mathcal{G}(\mathcal{I} + \mathcal{S}_2)^{-1}\mathcal{A}_{00}^{-1}(\mathcal{I} + \mathcal{S}_1)^{-1}(\mathcal{I} + \mathcal{S}_2)^{-1} \in \mathfrak{S}_\infty(\mathcal{H}^2). \tag{75}$$

(75) follows from below:

$$\mathcal{G}(\mathcal{I}+\mathcal{S}_2)^{-1}\mathcal{A}_{00}^{-1} =$$

$$= \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -iF_a^{1/2}Q_{a,b}^* \\ 0 & I \end{pmatrix} \begin{pmatrix} F_a^{-1} & 0 \\ 0 & (aI+Q_{a,b}Q_{a,b}^*)^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} GF_a^{-1} & -GF_a^{-1/2}Q_{a,b}^*(aI+Q_{a,b}Q_{a,b}^*)^{-1} \\ 0 & 0 \end{pmatrix} \in \mathfrak{S}_\infty(\mathcal{H}^2), \quad (76)$$

because $GF_a^{-1/2} \in \mathfrak{S}_\infty(\mathcal{H}) \quad \Longrightarrow \quad GF_a^{-1} \in \mathfrak{S}_\infty(\mathcal{H})$, others oper. in (76) are bounded.

$\Longrightarrow$ (look Th. 7.2, [15]) for unbounded $\mathcal{G} := \operatorname{diag}(G; 0)$:

a) (69) — an abstr. parab. eq.,

b) in Cauchy pr. (69) oper. coeff. $\mathcal{B}$ is a gener. of a semigroup that analit. in a sector $\supset (t>0)$.

Repeat Th. 1.4 from (69) ... □

The complex case: G is not subordinated to $F_a^{1/2}$, but is subordinated to F (or $F_a \gg 0$).

Theorem 1.6

Let in problem (11) the conditions

$$B \geqslant \gamma_B I, \quad B_b := B + bI \geqslant \alpha_B I, \quad \alpha_B > 0,$$

$$F \geqslant \gamma_F I,\ F_a := F + aI \geqslant \alpha_F I,\ \alpha_F > 0,\ \mathcal{D}(G) \supset \mathcal{D}(F_a) = \mathcal{D}(F),$$
$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(F), \quad f(t) \in W_p^1([0,T];\mathcal{H}), \quad p > 1,$$

be satisfied, and tech. cond.

$$(I + GF_a^{-1})^{-1} \in \mathcal{L}(\mathcal{H}). \tag{77}$$

holds true. Then this problem has a uniq. s.s. on $[0,T]$.

Ideas of proof:

Repeat trans. from (11) to (53): $\dfrac{dz}{dt} + \mathcal{A}_{a,b} z = f_{a,u}(t), \quad z(0) = y^0.$
For oper. matrice $\mathcal{A}_{a,b}$:

$$\mathcal{A}_{a,b} = \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I + iF_a^{-1/2} G F_a^{-1/2} & iQ_{a,b}^{+} \\ iQ_{a,b} & aI \end{pmatrix} \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix}, \mathcal{D}(\mathcal{A}_{a,b}) = \mathcal{D}(F) \oplus \mathcal{D}(B_b^{1/2}). \tag{78}$$

(78) def. correct. on $\mathcal{D}(\mathcal{A}_{a,b})$, because of: $\mathcal{D}(F) \subset \mathcal{D}(G) \quad \Longrightarrow$
$z_1 \in \mathcal{D}(F) = \mathcal{D}(F_a)$, $Gz_1 \in \mathcal{H}$, $z = (z_1; z_2)^\tau \in \mathcal{D}(\mathcal{A}_{a,b})$.

Let's show that oper. (from (78))

$$\mathcal{G}_{a,b} := \begin{pmatrix} I + iF_a^{-1/2}GF_a^{-1/2} & iQ_{a,b}^{+} \\ iQ_{a,b} & aI \end{pmatrix} \tag{79}$$

is bounded and dense defined. For this consider

$$G = J_G|G| = |G|^{1/2}J_G|G|^{1/2}, \;\; |G| = (G^2)^{1/2} \geqslant 0, \;\; J_G = J_G^* \in \mathcal{L}(\mathcal{H}), \;\; \mathcal{D}(G) = \mathcal{D}(|G|).$$

$\Longrightarrow$ dense def. oper. $F_a^{-1/2}GF_a^{-1/2}$ may be written in the form

$$F_a^{-1/2}GF_a^{-1/2} = (F_a^{-1/2}|G|^{1/2})J_G(|G|^{1/2}F_a^{-1/2}) =: V_a^{+}J_GV_a, \tag{80}$$
$$V_a := |G|^{1/2}F_a^{-1/2}, \; V_a^{+} := F_a^{-1/2}|G|^{1/2}, \; \mathcal{D}(V_a^{+}) := \mathcal{D}(|G|^{1/2}),$$

and

$$V_a \in \mathcal{L}(\mathcal{H}), \quad \mathcal{D}(V_a^{+}) = \mathcal{D}(|G|^{1/2}) \supset \mathcal{D}(|G|) = \mathcal{D}(G), \quad V_a^{+} = V_a^{*}|\mathcal{D}(|G|^{1/2}).$$

(look proof of Lemma 1.2, substitution $B \mapsto B_b$).

$\Longrightarrow$ $F_a^{-1/2}GF_a^{-1/2}$ is bounded on dense **set** $\in \mathcal{H}$ and $\overline{F_a^{-1/2}GF_a^{-1/2}} = V_a^{*}J_GV_a$.

$\Longrightarrow$ $\mathcal{G}_{a,b}$ from (79) is bounded and dense defined,

$$\overline{\mathcal{G}_{a,b}} := \begin{pmatrix} I + iV_a^{*}J_GV_a & iQ_{a,b}^{*} \\ iQ_{a,b} & aI \end{pmatrix} \quad \Longrightarrow$$

$$\mathrm{Re}(\overline{\mathcal{G}_{a,b}}z, z)_{\mathcal{H}^2} \geqslant \|z_1\|_{\mathcal{H}}^2 + a\|z_2\|_{\mathcal{H}}^2 \geqslant (\min(1;a))\|z\|_{\mathcal{H}^2}^2, \quad z = (z_1; z_2)^{\tau} \in \mathcal{H}^2. \tag{81}$$

As in Th. 1.1

$$\mathcal{A} := \overline{\mathcal{A}_{a,b}} = \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I + iV_a^* J_G V_a & iQ_{a,b}^* \\ iQ_{a,b} & aI \end{pmatrix} \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix}, \qquad (82)$$

$$\mathcal{D}(\mathcal{A}) = \Big\{ z = (z_1; z_2)^\tau \in \mathcal{H}^2 : \ z_1 + iF_a^{-1/2} V_a^* J_G V_a F_a^{-1/2} z_1 + iF_a^{-1/2} Q_{a,b}^* z_2 \in \mathcal{D}(F_a) \Big\},$$

and

$$\mathcal{A}z = \begin{pmatrix} F_a (z_1 + iF_a^{-1/2} V_a^* J_G V_a F_a^{-1/2} z_1 + iF_a^{-1/2} Q_{a,b}^* z_2) \\ iQ_{a,b} F_a^{1/2} z_1 + a z_2 \end{pmatrix}, \quad z \in \mathcal{D}(\mathcal{A}). \qquad (83)$$

Using property (81):

$$\mathrm{Re}(\mathcal{A}z, z)_{\mathcal{H}^2} \geqslant \|F_a^{1/2} z_1\|_{\mathcal{H}}^2 + a\|z_2\|_{\mathcal{H}}^2 \geqslant (\min(\alpha_F; a))\|z\|_{\mathcal{H}^2}^2, \ z \in \mathcal{D}(\mathcal{A}) \quad \Longrightarrow$$

$\mathcal{A} = \overline{\mathcal{A}_{a,b}}$ is a max. uniformly accr. oper. $\Longrightarrow$ use the scheme of proof Th. 1.3.
Namely, consider Cauchy Problem

$$\frac{dz}{dt} + \mathcal{A}z = f_{a,u}(t), \quad z(0) = y^0, \qquad (84)$$

$$f_{a,u}(t) := \mathrm{e}^{-at}(f(t) + bu(t); 0)^\tau =: f_{a,0}(t) + b\mathrm{e}^{-at}(u(t); 0)^\tau,$$

use the steps 1)– 3) of proof of Th. 1.3 $\Longrightarrow$ (84) will have a uniq. s.s. $z(t)$ on $[0, T]$.
It means that $z(t) = (z_1(t); z_2(t))^\tau$ is a contin. diff. solution of the system:

$$\frac{dz_1}{dt} + F_a\Big(z_1(t) + iF_a^{-1/2}V_a^* J_G V_a F_a^{1/2} z_1(t) + iF_a^{-1/2} Q_{a,b}^* z_2(t)\Big) =$$

$$= \mathrm{e}^{-at}(f(t) + bu(t)), \quad z_1(0) = u^1, \quad t \in [0, T], \tag{85}$$

$$\frac{dz_2}{dt} + az_2(t) + iQ_{a,b}F_a^{1/2} z_1(t) = 0, \quad z_2(0) = -iB_b^{1/2}u^0. \tag{86}$$

$$\Longrightarrow \quad z_2(t) = -i\mathrm{e}^{-at}B_b^{1/2}u^0 - i\int_0^t \mathrm{e}^{-a(t-s)} Q_{a,b} F_a^{1/2} z_1(s)ds.$$

Let's put $z_2(t)$ into (85) $\Longrightarrow$

$$\frac{dz_1}{dt} + F_a\Big(z_1(t) + iF_a^{-1/2}V_a^* J_G V_a F_a^{1/2} z_1(t) + F_a^{-1/2} Q_{a,b}^* B_b^{1/2} u^0 \mathrm{e}^{-at} +$$

$$+ F_a^{-1/2} Q_{a,b}^* \int_0^t \mathrm{e}^{-a(t-s)} Q_{a,b} F_a^{1/2} z_1(s)ds\Big) = \mathrm{e}^{-at}(f(t) + bu(t)), \quad z_1(0) = u^1. \tag{87}$$

Here $\varphi(t) := z_1(t) + iF_a^{-1/2}V_a^* J_G V_a F_a^{1/2} z_1(t) + F_a^{-1/2} Q_{a,b}^* B_b^{1/2} u^0 \mathrm{e}^{-at} +$

$$+ F_a^{-1/2} Q_{a,b}^* \int_0^t \mathrm{e}^{-a(t-s)} Q_{a,b} F_a^{1/2} z_1(s)ds, \quad t \in [0, T], \tag{88}$$

is contin. on t with values in $\mathcal{D}(F_a) = \mathcal{D}(F)$, because of $z(t) \in \mathcal{D}(\mathcal{A})$.

$$\varphi_0(t) \ := \ F_a^{-1/2}Q_{a,b}^* B_b^{1/2}u^0\mathrm{e}^{-at} \ = \ F_a^{-1/2}Q_{a,b}^+ B_b^{1/2}u^0\mathrm{e}^{-at} \ = \ F_a^{-1}B_b u^0\mathrm{e}^{-at},$$

has the same property, because $u^0 \in \mathcal{D}(B) = \mathcal{D}(B_b) \implies$ (88) may be written in the form

$$z_1(t) + iF_a^{-1/2}V_a^* J_G V_a F_a^{1/2} z_1(t) + F_a^{-1/2}Q_{a,b}^* \int_0^t \mathrm{e}^{-a(t-s)}Q_{a,b}F_a^{1/2}z_1(s)ds =$$
$$= \varphi_1(t) := \varphi(t) - \varphi_0(t), \quad t \in [0,T], \quad (89)$$

$$\varphi_1(t) \in C([0,T];\mathcal{D}(F_a)) \implies \varphi_1(t) = F_a^{-1}\eta(t), \quad \eta(t) \in C([0,T];\mathcal{H}).$$

Let's consider **"associated" equation** (with(89), subst. $z_1(t) = F_a^{-1}\psi(t)$, formal acting from left by F_a):

$$\psi(t) + iF_a^{1/2}V_a^* J_G V_a F_a^{-1/2}\psi(t) +$$
$$+ \int_0^t \mathrm{e}^{-a(t-s)}F_a^{1/2}Q_{a,b}^* Q_{a,b}F_a^{-1/2}\psi(s)ds = \eta(t), \quad t \in [0,T], \quad (90)$$

Here $F_a^{1/2}Q_{a,b}^*Q_{a,b}F_a^{-1/2} = F_a^{1/2}Q_{a,b}^*B_b^{1/2}F_a^{-1} = F_a^{1/2}Q_{a,b}^+B_b^{1/2}F_a^{-1} = B_bF_a^{-1}.$ (91)

$\mathcal{D}(F) \subset \mathcal{D}(G) \implies |G|^{1/2}F_a^{-1}\psi \in \mathcal{D}(|G|^{1/2}) = \mathcal{D}(J_G|G|^{1/2}) = \mathcal{D}(|G|^{1/2}J_G) \implies$

$$\forall\,\psi \in \mathcal{H}, \quad F_a^{1/2}V_a^*J_GV_aF_a^{-1/2}\psi = F_a^{1/2}V_a^*J_G|G|^{1/2}F_a^{-1}\psi =$$
$$= F_a^{1/2}V_a^+J_G|G|^{1/2}F_a^{-1}\psi = |G|^{1/2}J_G|G|^{1/2}F_a^{-1}\psi = GF_a^{-1}\psi. \quad (92)$$

$\implies$ eq. (90) is written in the form:

$$\psi(t) + iGF_a^{-1}\psi(t) + \int_0^t \mathrm{e}^{-a(t-s)}B_bF_a^{-1}\psi(s)ds = \eta(t), \quad t \in [0,T]. \quad (93)$$

GF_a^{-1}, $B_bF_a^{-1}$ — bound., $I + iGF_a^{-1}$ — bound. invers. $\implies$ (93) $\sim$ Volterra integr. eq.

$$\psi(t) + \int_0^t \mathrm{e}^{-a(t-s)}(I + iGF_a^{-1})^{-1}B_bF_a^{-1}\psi(s)ds = (I + iGF_a^{-1})^{-1}\eta(t), \quad (94)$$

with kernel $\mathrm{e}^{-a(t-s)}(I + iGF_a^{-1})^{-1}B_bF_a^{-1}$, that is contin. on t,s oper.–function with values in $\mathcal{L}(\mathcal{H})$. $(I + iGF_a^{-1})^{-1}\eta(t)$ — a known contin. function on $t \in [0,T]$ $\implies$ (94) and (93) have a uniq. solution $\psi(t) \in C([0,T];\mathcal{H})$.

Substitution $\psi(t) = F_az_1(t)$, $z_1(t) \in C([0,T];\mathcal{D}(F_a))$, use (91)–(92) $\implies$

$$F_a\Big(z_1(t)+iF_a^{-1/2}V_a^*J_GV_aF_a^{1/2}z_1(t)+F_a^{-1/2}Q_{a,b}^*\int_0^t e^{-a(t-s)}Q_{a,b}F_a^{1/2}z_1(s)ds-\varphi_1(t)\Big)=0.$$

$\Longrightarrow$ integr. eq. (89) has a uniq. s.s. $z_1(t)\in C([0,T];\mathcal{D}(F_a))$.
From (91) and from

$$F_a^{-1/2}Q_{a,b}^*Q_{a,b}F_a^{1/2}z_1(s)=F_a^{-1}\Big(F_a^{1/2}Q_{a,b}^*Q_{a,b}F_a^{-1/2}\Big)F_az_1(s)=$$
$$=F_a^{-1}B_bF_a^{-1}F_az_1(s)=F_a^{-1}B_bz_1(s)\in C([0,T];\mathcal{D}(F_a))\quad\Longrightarrow$$

that in (89) every item $\in C([0,T];\mathcal{D}(F_a))$.
At the same time we get

$$F_a^{1/2}V_a^*J_GV_aF_a^{1/2}z_1(t)\;=\;Gz_1(t),\;\;F_a^{1/2}Q_{a,b}^*B_b^{1/2}u^0\;=\;B_bu^0\qquad(u^0\;\in\;\mathcal{D}(B)),$$

$$\int_0^t e^{-a(t-s)}F_a^{1/2}Q_{a,b}^*Q_{a,b}F_a^{1/2}z_1(s)ds\;\;=\;\;e^{-at}\int_0^t e^{as}B_bz_1(s)ds,\qquad t\;\in\;[0,T].$$

Use in integr. eq. (87): $z_1(t)=e^{-at}(du(t)/dt)$, $\quad f_{a,u}(t)=\;e^{-at}(f(t)+bu(t))\;\;\Longrightarrow$
$u(t)$ is s.s. of Cauchy Problem (11): $\quad d^2u/dt^2+(F-aI+iG)\frac{du}{dt}+B_bu=f(t)+bu(t).$

Remark 1.1

From proof of Th. 1.6 it is seen that its propos. are true, if eq. (93) has uniq. solution $\psi(t) \in C([0,T];\mathcal{H})$, and cond. (77) may be not satisfied.

Remark 1.2

$I + iGF_a^{-1}$ is invertible in $\mathcal{H}$, may be, with an unbounded inverse operator.
Actually, if $u \in \mathcal{H}$ is an element for which the cond. $(I + iGF_a^{-1})u = 0$ is satisfied, then

$$(F_a^{-1}u, u)_{\mathcal{H}} + i(F_a^{-1}GF_a^{-1}u, u)_{\mathcal{H}} = 0,$$

$\Longrightarrow \quad u = 0,$ because $F_a^{-1} > 0$ and $G = G^*$.

Corollary 1.4

$(I + iGF_a^{-1})$ has bounded inverse operator in every of following cases :
1°. $\|GF_a^{-1}\| < 1$ or the condition for spectral radius $r(GF_a^{-1})$ of operator GF_a^{-1} is satisfied: $r(GF_a^{-1}) < 1$;
2°. operator GF_a^{-1} is compact;
3°. operators G and F are commutative ones.

Ideas of proof:
1°. The first propos. is evidently, because in this case bounded inverse oper. exists and it is represented by the Neumann series

$$(I+iGF_a^{-1})^{-1} = \sum_{k=0}^{\infty}(iGF_a^{-1})^k, \quad \|(I+iGF_a^{-1})^{-1}\| \leqslant 1/(1-\|GF_a^{-1}\|), \quad \|GF_a^{-1}\| < 1.$$

2°. If GF_a^{-1} is compact, then inverse oper. $I + iGF_a^{-1}$ (by Th. 1.3) has bounded inverse (Th. Fred.).
3°. If G and F are commutative, then bounded oper. GF_a^{-1} is self-adjoint and may be written as $GF_a^{-1} = V_a J_G V_a$ (here $V_a = |G|^{1/2} F_a^{-1/2} = V_a^*$, $J_G = J_G^*$, $J_G = 1$). Therefore,

$$\mathrm{Re}((I \pm iGF_a^{-1})u, u)_{\mathcal{H}} = \mathrm{Re}((I \pm iV_a J_G V_a u, u)_{\mathcal{H}} = \|u\|^2_{\mathcal{H}}, \quad \forall u \in \mathcal{H},$$

$\Longrightarrow$ $I + iGF_a^{-1}$ is an uniformly accr. $\Longrightarrow$ $\exists$ bounded inverse oper., $\|(I + iGF_a^{-1})^{-1}\| \leqslant 1$.

Remark 1.3

If in (11) the cond. $u^0 \in \mathcal{D}(B)$ to replace by stronger cond. $u^0 \in \mathcal{D}(F) \subset \mathcal{D}(B)$, then s.s. to (11) $\in C([0,T];\mathcal{D}(F)) \subset C([0,T];\mathcal{D}(B))$ (its existence and uniq. were proved in Th. 1.2 – Th. 1.6).
Actually, $F(du/dt) \in C([0,T];\mathcal{H})$ accord. to Def. 1.1 of s.s. $\Longrightarrow$

$$\int_0^t F\frac{du}{d\xi}\,d\xi = F\int_0^t \frac{du}{d\xi}\,d\xi = F(u(t)-u(0)) \in C([0,T];\mathcal{H}),$$

and if $u(0)=u^0 \in \mathcal{D}(F) \quad \Longrightarrow \quad Fu(t) \in C([0,T];\mathcal{H})$.

Remark 1.4

Let $G=0$. If the cond.

$$u^0 \in \mathcal{D}(F), \quad u^1 \in \mathcal{D}(F), \quad f(t) \in W_p^1([0,T];\mathcal{H}), \quad p>1,$$

be satisfied, then (11) is stable at any substitution of an unknown function by the formula

$$u(t) = \mathrm{e}^{ct}v(t), \quad c>0. \tag{95}$$

It means that after this subs. of s.s. $u(t)$ to (11) trans. to s.s. $v(t)$ to the trans. Cauchy Problem. If only the cond. $u^0 \in \mathcal{D}(B) \supset \mathcal{D}(F)$ is satisfied, then after the subs. (95) of s.s. $u(t)$ to (11) may be trans. to $v(t)$, that is not s.s. accord. to Def. 1.1.

On the base of proved above propos. on the solvability of Cauchy Problem (11) ("shortened" problem (9)), consider a problem of the solvability

$$\frac{d^2u}{dt^2} + (F+iG)\frac{du}{dt} + Bu + \sum_{k=1}^{m}\int_0^t G_k(t,s)A_k u(s)ds = f(t),$$

$$u(0) = u^0,\ u'(0) = u^1,\ \mathcal{D}(F) \subset \mathcal{D}(B). \tag{96}$$

Our purpose — to formulate the restrict. on (unbounded) oper. coeff. A_k, $k = \overline{1,m}$, and the kernel function $G_k(t,s)$, which get us opportunity to prove the propos. on the solvability of (96) at the cond. that are the same for "shortened" problem: $A_k = 0$, $k = \overline{1,m}$.

Def. of s.s. to problem (96) for integro-diff. eq. is the same, that we have for "shortened" problem (Def. 1.1), with the appl.: in eq. (96) all items (and integr., too), must be contin. on t: $\in C([0,T];\mathcal{H})$.

Let's consider the simplest case for problem (96)

$$F = F^* \gg 0, \quad B = B^* \gg 0, \quad G \in \mathcal{L}(\mathcal{H}). \tag{97}$$

Let $u(t)$ is s.s. to (96), and for eq. (96) we act the same trans. were done for s.s. to "shortened" problem (11) in subsect. 1.2.

Trans. by formulas (13)–(21) $\Longrightarrow$ oper. matrice $\mathcal{A}_a$ from (21), that is an uniformly accr. and admit extension (closing) (Lemma 1.2, Th. 1.1), and accord. to Corol. 1.1 oper. matrice $(-\mathcal{A}) = (-\overline{\mathcal{A}_a})$ is a gener. of a contracting semigroup $\mathcal{U}(t)$, $\|\mathcal{U}(t)\| \leqslant e^{-ta}$, $t \geqslant 0$

Problem (96) trans. to the Cauchy Problem

$$\frac{dz}{dt} + \mathcal{A}_a z + \begin{pmatrix} \sum_{k=1}^{m} \int_0^t e^{-a(t-\xi)} V_k(t,\xi) A_k z_1(\xi) ds \\ 0 \end{pmatrix} = \widehat{f}_a(t), \quad z(0) = z^0,$$

$$z(t) = e^{-at} y(t) = e^{-at}(du/dt; dv/dt)^{\tau}, \ \ dv/dt := -iB^{1/2}u(t), \ \ v(0) = 0,$$

$$\widehat{f}_a(t) := (e^{-at}(f(t) - \varphi_0(t)); 0)^{\tau}, \quad \varphi_0(t) := \sum_{k=1}^{m} \int_0^t G_k(t,s) A_k u^0 ds,$$

$$V_k(t,\xi) := \int_\xi^t G_k(t,s) ds, \ k = \overline{1,m}, \ \mathcal{D}(\mathcal{A}_a) = \mathcal{D}(F_a) \oplus \mathcal{D}(B^{1/2}), \quad (98)$$

after closing $\mathcal{A}_a$ to the problem:

$$\frac{dz}{dt} + \mathcal{A} z + \sum_{k=1}^{m} \int_0^t e^{-a(t-\xi)} \mathcal{V}_k(t,\xi) \mathcal{A}_k z(\xi) ds = \widehat{f}_a(t), \quad z(0) = z^0, \quad (99)$$

$$\mathcal{V}_k(t,\xi) := \operatorname{diag}(V_k(t,\xi); 0), \quad \mathcal{A}_k := \operatorname{diag}(A_k; 0), \quad k = \overline{1,m}. \quad (100)$$

Theorem 1.7

Let in problem (96) cond. (97) and the cond.

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(F), \tag{101}$$

and one of the cond.

$$f(t) \in W_p^1([0,T];\mathcal{H}), \quad p > 1, \tag{102}$$

$$F^{-1} \in \mathfrak{S}_\infty(\mathcal{H}), \quad f(t) \in C^\alpha([0,T];\mathcal{H}), \quad 0 < \alpha \leqslant 1. \tag{103}$$

be satisfied.
Moreover, let the restrictions

$$G_k(t,s), \ \ \partial G_k(t,s)/\partial t \in C(\triangle_T;\mathcal{L}(\mathcal{H})),$$
$$\triangle_T := \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\}, \quad k = \overline{1,m}, \tag{104}$$

be satisfied.
Let's require

$$\mathcal{D}(A_k) \supset \mathcal{D}(F^{1/2}), \quad k = \overline{1,m}. \tag{105}$$

Then problem (99), (100) has a uniq. s.s. on $[0,T]$.

Ideas of proof:

Let properties (97), (101)–(105) be satisfied for problem (99), (100).

Cond. (97) and $\mathcal{D}(F) \subset \mathcal{D}(B) \implies$ oper. $(-\mathcal{A})$ in (99) is a gener. of a contracting semigroup.

The first cond. in (103) $\implies$ oper. $(-\mathcal{A})$ is a gener. of an analit. semigroup that given in some sector $\supset (t > 0)$ (Th. 1.2 and Th. 1.4).

Cond. (101) $\implies$ in problem (99)

$$z^0 = y^0 = (u^1; -iB^{1/2}u^0)^\tau \in \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A}_a) \subset \mathcal{D}(\mathcal{A}), \qquad (106)$$

is satisfied.

Cond. (104) $\implies$ the given function $\widehat{f}_a(t)$ (def. (98)) $\implies$ $\widehat{f}_a(t) \in W_p^1([0,T];\mathcal{H}^2)$, $p > 1$), (if $f(t) \in W_p^1([0,T];\mathcal{H})$), and $\widehat{f}_a(t) \in C^\alpha([0,T];\mathcal{H}^2)$ (if $f(t) \in C^\alpha([0,T];\mathcal{H})$, $0 < \alpha \leqslant 1$), because of

$$\varphi_0(t) := \sum_{k=1}^{m} \int_0^t G_k(t,s) A_k u^0 ds \in C^1([0,T];\mathcal{H}), \quad u^0 \in \mathcal{D}(B). \qquad (107)$$

Let's prove that in (99) the cond.

$$\mathcal{V}_k(t,s), \ \ \partial\mathcal{V}_k(t,s)/\partial t \in SC(\triangle_T;\mathcal{L}(\mathcal{H}^2)), \quad k=\overline{1,m},$$

$$\mathcal{D}(\mathcal{A}_k) \supset \mathcal{D}(\mathcal{A}), \quad k=\overline{1,m},$$

be satisfied.

Actually, cond. (107) directly follows from (104) and first formulas (100).
Evidently, that

$$\mathcal{D}(\mathcal{A}_k) = \mathcal{D}(A_k) \oplus \mathcal{H}, \quad k=\overline{1,m}.$$

Accordingly to Th. 1.1

$$\mathcal{D}(\mathcal{A}) = \{z=(z_1,z_2)^\tau : z_1 \in \mathcal{D}(F_a^{1/2}), \quad z_1 + iF_a^{-1/2}Q_a^* z_2 \in \mathcal{D}(F_a)\} \subset \\ \subset \mathcal{D}(F_a^{1/2}) \oplus \mathcal{H} \subset \mathcal{D}(\mathcal{A}_k), \quad k=\overline{1,m}. \quad (108)$$

$\Longrightarrow$ for problem (99) has a uniq. s.s. $z(t)$ on $[0,T]$. □

From Th. 1.7 follows the proposition.

Theorem 1.8

If the cond. of Th. 1.7 are satisfied, then Cauchy Problem (96) for the second order integro-diff. eq. has a uniq. s.s. on $[0, T]$.

Ideas of proof:
Cond. (97), (101)–(105) $\implies$ according to Th. 1.7 problem (99) has a uniq. s.s. on $[0, T]$ $\implies$ eq. (99) and initial cond. are satisfied.
And all items in eq. (99) are contin. functions on t with values in $\mathcal{H}$, $t \in [0, T]$.
It is means that

$$\frac{dz_1}{dt} + F_a(z_1(t) + iF_a^{-1/2}Q_a^* z_1(t)) + iGz_1(t) + \\ + \sum_{k=1}^{m} \int_0^t \mathrm{e}^{-a(t-\xi)} V_k(t,\xi) A_k z_1(\xi) d\xi = \mathrm{e}^{-at}(f(t) - \varphi_0(t)), \quad z_1(0) = u^1, \quad (109)$$

$$\frac{dz_2}{dt} + az_2(t) + iQ_a F_a^{1/2} z_1(t)) = 0, \quad z_2(0) = -iB^{1/2}u^0, \quad (110)$$

here all items $\in C([0, T]; \mathcal{H})$.

Here, in particularly, $z_1(t) \in \mathcal{D}(F_a^{1/2}) = \mathcal{D}(F^{1/2}) \supset \mathcal{D}(A_k) \quad \Longrightarrow$

$A_k z_1(t)$ is a contin. function with values in $\mathcal{H} \quad \Longrightarrow \quad$ integr. items are contin. functions on t, too.

Repeat:

parts of proof Th. 1.2, that connected with the considering of Volterra integr. eq. (35), (36) and connecting with step 2) of proof Th. 1.2 for (102);

step 3) for (103).

$\Longrightarrow \quad z_1(t) \in C([0,T];\mathcal{H}(F_a)$.

$\Longrightarrow$ from (109) and exception $z_2(t)$ from (109), (110) and using $u(t)$, $du/dt \equiv \mathrm{e}^{-at} z_1(t)$, $u(0) = u^0$

$\Longrightarrow$ Cauchy Problem (96) has a uniq. s.s. on $[0,T]$. □