Complete Second Order Volterra Integro-Differential Equations in Hilbert Space (Lection II)

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Now we study Cauchy Problem

$$\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + Bu + \sum_{k=1}^{m} \int_0^t G_k(t,s) A_k u(s) ds = f(t), \quad (1)$$
$$u(0) = u^0, \quad u'(0) = u^1,$$

and "shortened" problem for diff. eq.

$$\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + Bu = f(t), \quad u(0) = u^0, \quad u'(0) = u^1. \quad (2)$$

Assumption: B is dominating.

If $F = G = 0$, $A_k = 0$ $(k = \overline{1, m})$ $\implies$ (1) is an abstract. hyperb. eq.

Let in problem (2) the cond.

$$G = 0, \quad F = F^* \geqslant 0, \quad B = B^* \gg 0, \tag{3}$$

$$\mathcal{D}(B^{1/2}) \subset \mathcal{D}(F) \subset \mathcal{H}, \tag{4}$$

be satisfied.

Definition 2.1

$u = u(t) \in \mathcal{H}$ is a strong solution to (2)–(4) $\quad t \in [0, T] \quad$ if:
a) all items in (2) and the function $B^{1/2}(du/dt) \in C([0, T]; \mathcal{H})$;
b) eq. (2) is satisfied, $\quad t \in [0, T]$;
c) init. cond. (2) are satisfied.

Remark 2.1

From Def. 2.1 $\quad \Longrightarrow \quad$ the necessary cond. for $\exists$ s.s:

$$u^0 \in \mathcal{D}(B), \;\; u^1 \in \mathcal{D}(B^{1/2}), \;\; f(t) \in C^1([0, T]; \mathcal{H}).$$

Let problem (2)–(4) has s.s. on $[0, T]$.
Introduce new unknown function $v(t)$ by the formula (as in S. 1.2, Lect. I):

$$\frac{dv}{dt} = -iB^{1/2}u(t), \quad v(0) = 0. \tag{5}$$

From Def 2.1 $\Longrightarrow$ $B^{1/2}u(t) \in C^1([0, T]; \mathcal{H})$, $t \in [0, T]$ $\Longrightarrow$ $v(t) \in C^2([0, T]; \mathcal{H})$, $t \in [0, T]$, and

$$\frac{d^2v}{dt^2} + iB^{1/2}\frac{du}{dt} = 0, \quad \frac{dv}{dt}(0) = -iB^{1/2}u^0. \tag{6}$$

Use (5), (6) $\Longrightarrow$ problem (2)–(4) trans. to Cauchy Problem for first order diff. eq. in $\mathcal{H}^2$:

$$\frac{dy}{dt} + \mathcal{A}_0 y = f_0(t), \quad y(0) = y^0, \tag{7}$$

$$y(t) := \left(\frac{du}{dt}; \frac{dv}{dt}\right)^\tau, \quad y^0 := (u^1; -iB^{1/2}u^0)^\tau, \quad f_0(t) := (f(t); 0)^\tau,$$

$$\mathcal{A}_0 := \begin{pmatrix} F & iB^{1/2} \\ iB^{1/2} & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(B^{1/2}) \oplus \mathcal{D}(B^{1/2}). \tag{8}$$

Here $\mathcal{A}_0$ defined on $\mathcal{D}(\mathcal{A}_0)$ correctly, because of $\mathcal{D}(B^{1/2}) \subset \mathcal{D}(F)$.

Lemma 2.1

$\mathcal{A}_0$, that defined on $\mathcal{D}(\mathcal{A}_0)$, is max. accr. operator.

Ideas of proof: The accr. property of $\mathcal{A}_0$ follows from:

$$\mathrm{Re}(\mathcal{A}_0 y, y)_{\mathcal{H}^2} = (F y_1, y_1)_{\mathcal{H}} \geqslant 0, \quad \forall y = (y_1; y_2)^{\tau} \in \mathcal{D}(\mathcal{A}_0).$$

Use the factorization:

$$\mathcal{A}_0 := i \begin{pmatrix} I & iFB^{-1/2} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & B^{1/2} \\ B^{1/2} & 0 \end{pmatrix}.$$

Here:
(4) $\Longrightarrow$ $FB^{-1/2}$ is bounded $\Longrightarrow$ the first factor has bounded inverse operator.
$B^{1/2} \gg 0$ $\Longrightarrow$ the second factor has bounded inverse operator, too.
$\Longrightarrow$ $\mathcal{A}_0$ is closed on $\mathcal{D}(\mathcal{A}_0)$ and $\mathcal{R}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A}_0^{-1}) = \mathcal{H}^2$.
$\mathcal{A}_0$ is accr. operator $\Longrightarrow$ $\mathcal{A}_0$ is max. accr. operator. □

Corollary 2.1

$(-\mathcal{A}_0)$ is a gener. of a contracting (C_0)–semigroup.

Theorem 2.1

Let in Cauchy Problem (2)–(4) the conditions

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(B^{1/2}), \quad f(t) \in W_p^1([0,T];\mathcal{H}), \quad p > 1, \tag{9}$$

be satisfied.
$\Longrightarrow$ $\exists$ a uniq. s.s. (Def. 2.1) on $[0,T]$.

Ideas of proof: Consider (7)–(8).
If cond. (9) are satisfied $\Longrightarrow$

$$y^0 = (u^1; -iB^{1/2}u^0)^\tau \in \mathcal{D}(B^{1/2}) \oplus \mathcal{D}(B^{1/2}) = \mathcal{D}(\mathcal{A}_0), \tag{10}$$

$$f_0(t) := (f(t); 0)^\tau \in W_p^1([0,T];\mathcal{H}^2), \quad p > 1. \tag{11}$$

$(-\mathcal{A}_0)$ — a gener. of a contracting (C_0)–semigroup; (10), (11) are satisfied $\Longrightarrow$ problem (7)–(8) (Th. S. Yakubov) has a uniq. s.s. $y(t)$ on $[0,T]$.
Return to $u(t)$, $v(t)$ $\Longrightarrow$

$$\frac{d^2u}{dt^2} + F\frac{du}{dt} + iB^{1/2}\frac{dv}{dt} = f(t), \quad \frac{du}{dt}(0) = u^1, \quad u(0) = u^0, \tag{12}$$

$$\frac{d^2v}{dt^2} + iB^{1/2}\frac{du}{dt} = 0, \quad \frac{dv}{dt}(0) = -iB^{1/2}u^0, \quad v(0) = 0. \tag{13}$$

Here:

All functions are from $C([0,T];\mathcal{H}$.

In particularly, (13) $\implies$ $B^{1/2}\dfrac{du}{dt} \in C([0,T];\mathcal{H})$.

Integrate (13) by t from 0 until t $\implies$ (5).

From (12) $\implies$ $-iB^{1/2}u(t) = \dfrac{dv}{dt} \in C([0,T];\mathcal{D}(B^{1/2}))$.

(dv/dt) $\to$ (12) $\implies$

$$\frac{d^2u}{dt^2} + F\frac{du}{dt} + Bu = f(t), \quad 0 \leqslant t \leqslant T, \tag{14}$$

and all items are contin. by t. □

Remark

From contin. $Bu(t)$ and $B^{1/2}\dfrac{du}{dt} =: i\dfrac{d^2v}{dt^2}$ $\implies$ initial cond. for eq. are satisfied in following sense:

$$\|B(u(t)-u^0)\| \to 0, \quad \left\|B^{1/2}\left(\frac{du}{dt} - u^1\right)\right\| \to 0 \quad (t \to +0). \tag{15}$$

Actually, because in (7)–(8) $y(t) \to y^0$, $t \to +0$ $\implies$

$$\frac{du}{dt} \to u^1 \quad (\text{в } \mathcal{H}), \quad \frac{dv}{dt} = -iB^{1/2}u(t) \to -iB^{1/2}u^0 \quad (\text{в } \mathcal{H}).$$

From this (using contin. $Bu(t)$ and $B^{1/2}(du/dt)$) $\implies$ (15).

Let the following cond. be satisfied:

$$G = 0, \quad F = F^* \geqslant \gamma_F I, \quad B = B^* \geqslant \gamma_B I, \quad \gamma_F, \gamma_B \in \mathbb{R}, \tag{16}$$

$$\mathcal{D}(F) \supset \mathcal{D}(B_b^{1/2}). \tag{17}$$

$\Longrightarrow$ Cauchy Problem arises:

$$\frac{d^2u}{dt^2} + F\frac{du}{dt} + Bu = f(t), \quad u(0) = u^0, \quad u'(0) = u^1. \tag{18}$$

Use the schemes S1.2, S2.1. $\Longrightarrow$ associated oper. matrice will have the same properties.

Represent oper. B in the form:

$$B = B_b - bI, \quad B_b := B + bI \geqslant \alpha_B I, \quad b + \gamma_B =: \alpha_B > 0, \tag{19}$$

and rewrite (18) in the form

$$\frac{d^2u}{dt^2} + F\frac{du}{dt} + B_b u = f_u(t) := f(t) + bu(t), \; u(0) = u^0, \; u'(0) = u^1.$$

Introduce a new uknown function $v(t)$ by the formulas:

$$\frac{dv}{dt} = -iB_b^{1/2}u(t), \quad v(0) = 0. \tag{20}$$

Then for s.s. to (16), (18) we have:

$$\frac{d^2v}{dt^2} + iB_b^{1/2}\frac{du}{dt} = 0, \quad v'(0) = -iB_b^{1/2}u^0, \quad v(0) = 0.$$

For $u(t)$ and $v(t)$ we get Cauchy problem

$$\frac{d^2u}{dt^2} + F\frac{du}{dt} + iB_b^{1/2}\frac{dv}{dt} = f_u(t), \quad u'(0) = u^1, \quad u(0) = u^0,$$

$$\frac{d^2v}{dt^2} + iB_b^{1/2}\frac{du}{dt} = 0, \quad v'(0) = -iB_b^{1/2}u^0, \quad v(0) = 0,$$

that may be written in the form

$$\frac{dy}{dt} + \mathcal{A}_{0,b}y = f_{0,u}(t), \quad y(0) = y^0, \tag{21}$$

$$y(t) := (u'(t); v'(t))^\tau, \quad y^0 := (u^1; -iB_b^{1/2}u^0)^\tau,$$
$$f_{0,u}(t) := (f_u(t); 0)^\tau, \quad f_u(t) := f(t) + bu(t),$$

$$\mathcal{A}_{0,b} := \begin{pmatrix} F & iB_b^{1/2} \\ iB_b^{1/2} & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_{0,b}) = \mathcal{D}(B_b^{1/2}) \oplus \mathcal{D}(B_b^{1/2}).$$

Oper. matrice $\mathcal{A}_{0,b}$ is defined correctly because of (17).

Consider problem (16)–(19).

The Def. of s.s. is the same, only we require replace the contin. property of $B^{1/2}(du/dt)$ by the analogical property for $B_b^{1/2}(du/dt)$.

Substitute in (21)

$$y(t) = \mathrm{e}^{at} z(t), \quad a > 0 \quad \Longrightarrow \tag{22}$$

for an unknown function $z(t)$ we get Cauchy Problem

$$\frac{dz}{dt} + \mathcal{A}_{a,b} z = f_{a,u}(t) := \mathrm{e}^{-at} f_{0,u}(t), \quad z(0) = y^0, \tag{23}$$

$$\mathcal{A}_{a,b} := \begin{pmatrix} F_a & iB_b^{1/2} \\ iB_b^{1/2} & aI \end{pmatrix}, \;\; F_a := F + aI, \;\; \mathcal{D}(\mathcal{A}_{a,b}) = \mathcal{D}(B_b^{1/2}) \oplus \mathcal{D}(B_b^{1/2}). \tag{24}$$

Choose constant $a > 0$ from the cond.

$$a + \gamma_F =: \alpha_F > 0 \quad \Longleftrightarrow \quad F_a \geqslant \alpha_F I \gg 0. \tag{25}$$

Lemma 2.2

If cond. (25) is satisfied $\Longrightarrow$ $\mathcal{A}_{a,b}$, that is defined on $\mathcal{D}(\mathcal{A}_{a,b})$ by formula (24) — a max. uniformly accr. operator:

$$Re\,(\mathcal{A}_{a,b}z, z)_{\mathcal{H}^2} \geqslant c\|z\|^2_{\mathcal{H}^2}, \quad c := (\min(\alpha_F; a)), \quad z \in \mathcal{D}(\mathcal{A}_{a,b}). \tag{26}$$

Ideas of proof:

(24), (25) $\Longrightarrow$ property (26) because of $\mathrm{Re}(\mathcal{A}_{a,b}z, z)_{\mathcal{H}^2} = (F_a z_1, z_1)_{\mathcal{H}} + a\|z_2\|^2_{\mathcal{H}}$.

From (26) $\Longrightarrow$ $\mathcal{A}_{a,b}$ has bounded inverse oper., that defined on $\mathcal{R}(\mathcal{A}_{a,b})$.

Let's prove that $\mathcal{R}(\mathcal{A}_{a,b}) = \mathcal{H}$ and get a formula for $\mathcal{A}^{-1}_{a,b}$.

For this consider

$$F_a u_1 + iB_b^{1/2}u_2 = v_1, \;\; iB_b^{1/2}u_1 + au_2 = v_2 \;\Longleftrightarrow\; \mathcal{A}_{a,b}u = v, \; v = (v_1; v_2)^\tau \in \mathcal{H}, \tag{27}$$

$u_1 \in \mathcal{D}(B_b^{1/2})$, $u_2 \in \mathcal{D}(B_b^{1/2}) \Longleftrightarrow u = (u_1; u_2)^\tau \in \mathcal{D}(\mathcal{A}_{a,b})$.
$u_1 \in \mathcal{D}(B_b^{1/2})$, (17) $\Longrightarrow$ $u_1 \in \mathcal{D}(F) = \mathcal{D}(F_a)$ $\Longrightarrow$ $F_a u_1 = v_1 - iB_b^{1/2}u_2$,
$u_2 = a^{-1}(v_2 - iB_b^{1/2}u_1)$.

$$B_b^{1/2} \gg 0 \Longrightarrow B_b^{-1/2}F_a u_1 = B_b^{-1/2}v_1 - iu_2 = B_b^{-1/2}v_1 - ia^{-1}(v_2 - iB_b^{1/2}u_1), \tag{28}$$

$$\Longrightarrow (B_b^{-1/2}F_a B_b^{-1/2} + a^{-1}I)u_1 = B_b^{-1/2}v_1 - ia^{-1}v_2. \tag{29}$$

Here:
(17), $F_a = F + aI \quad \Longrightarrow \quad F_a B_b^{-1/2}$ is bounded, $B_b^{-1/2} F_a B_b^{-1/2} \geqslant 0$ is self-adjoint oper., that is defined on all $\mathcal{H} \quad \Longrightarrow (a^{-1}I + \; B_b^{-1/2} F_a B_b^{-1/2} \gg 0)$ is self-adjoint, has bounded inverse oper. $T_{a,b} := \; (a^{-1}I + B_b^{-1/2} F_a B_b^{-1/2})^{-1} \gg 0$:

$$\|T_{a,b}\| = \|(a^{-1}I + B_b^{-1/2} F_a B_b^{-1/2})^{-1}\| \leqslant a. \tag{30}$$

From this and (29) $\quad \Longrightarrow$

$$\begin{aligned} u_1 &= B_b^{-1/2} T_{a,b} B_b^{-1/2} v_1 - ia^{-1} B_b^{-1/2} T_{a,b} v_2, \\ u_2 &= -ia^{-1} T_{a,b} B_b^{-1/2} v_1 + (a^{-1}I - a^{-2} T_{a,b}) v_2, \end{aligned} \tag{31}$$

$$\Longrightarrow \quad \mathcal{A}_{a,b}^{-1} = \begin{pmatrix} B_b^{-1/2} T_{a,b} B_b^{-1/2} & -ia^{-1} B_b^{-1/2} T_{a,b} \\ -ia^{-1} T_{a,b} B_b^{-1/2} & a^{-1} T_{a,b}^{1/2} B_b^{-1/2} F_a B_b^{-1/2} T_{a,b}^{1/2} \end{pmatrix}. \tag{32}$$

From above and (32) $\quad \Longrightarrow \quad \mathcal{A}_{a,b}^{-1}$ is bounded accr. oper., that is defined on all $\mathcal{H}^2$.

Let $\forall\, v_1$, $\forall\, v_2 \in \mathcal{H}$. $\qquad$ (31) $\quad \Longrightarrow \quad u_1 \in \mathcal{D}(B_b^{1/2})$, (27). Use inverse trans. in (31) and return to (27) $\quad \Longrightarrow \quad F_a u_1 \in \; \mathcal{H}$, $u_2 \in \mathcal{D}(B_b^{1/2})$ and (27) $\quad \Longrightarrow$

$$\mathcal{R}(\mathcal{A}_{a,b}^{-1}) = \mathcal{D}(\mathcal{A}_{a,b}) = \mathcal{D}(B_b^{1/2}) \oplus \mathcal{D}(B_b^{1/2}). \tag{33}$$

□

From Lemma 2.2 $\quad \Longrightarrow \quad (-\mathcal{A}_{a,b})$ — a gener. of a contracting semigroup $\mathcal{U}(t)$,

$$\|\mathcal{U}(t)\| \leqslant \mathrm{e}^{-c\,t}, \tag{34}$$

where $c > 0$ is constant from (26).

Theorem 2.2

Let in problem (18), (16), (17) the conditions

$$u^0 \in \mathcal{D}(B) = \mathcal{D}(B_b),\ u^1 \in \mathcal{D}(B_b^{1/2}),\ f(t) \in W_p^1([0,T];\mathcal{H}),\ p > 1, \tag{35}$$

be satisfied (B_b — from (19)). Then this problem has a uniq. s.s. on $[0,T]$.

Ideas of proof: Use the scheme of Th. 1.3.
Consider Cauchy Problem (23), (24). Let cond. (35) be satisfied.
As in Th. 1.3 we have (35) $\Longrightarrow$ $y^0 = (u^1; -iB_b^{1/2}u^0) \in \mathcal{D}(\mathcal{A}_{a,b})$.
$f_{a,u}(t) \in W_p^1([0,T];\mathcal{H}^2)$, $p > 1$, $\Longrightarrow$ by Th. 2.1 Cauchy Pr. (23), (24) has s.s. $z(t)$:

$$z(t) = \mathcal{U}(t)y^0 + \int_0^t \mathcal{U}(t-s)f_{a,u}(s)ds, \quad t \in [0,T], \tag{36}$$

here $\mathcal{U}(t)$ — a contracting semigroup that corresponds to $(-\mathcal{A}_{a,b})$ with estimation (34).
For $f_{a,u}(t)$ we have:

$$f_{a,u}(t) = f_{a,b}(t) + b\mathrm{e}^{-at}\int_0^t \mathrm{e}^{a\xi}P_1 z(\xi)d\xi, \qquad P_1 \textit{ is orthopr. from } \mathcal{H}^2 \textit{ to } \mathcal{H}. \tag{37}$$

From (36), (37) $\Longrightarrow$ integr. eq. Repeat step 3) from the proof of Th. 1.3. ...
At last: if (35) $\Longrightarrow$ problem (23), (24) has uniq. s.s. on $[0,T]$.
Repeat the inverse trans. and return from (23) to initial problem (18), (16), (17).

Assumptions:

$$G = G^* \neq 0, \quad F = F^* \geqslant \gamma_F I, \quad B = B^* \geqslant \gamma_B I, \quad \gamma_F, \gamma_B \in \mathbb{R}. \tag{38}$$

Use the approach from S. 2.2 ($G = 0$) to consider Cauchy Problem

$$\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + Bu = f(t), \quad u(0) = u^0, \quad u'(0) = u^1. \tag{39}$$

Introduce $B_b := B + bI$, $b + \gamma_b =: \alpha_B > 0 \quad \Longrightarrow$

$$\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + B_b u = f(t) + bu(t), \quad u(0) = u^0, \quad u'(0) = u^1. \tag{40}$$

Let the following cond. be satisfied:

$$\mathcal{D}(B_b^{1/2}) \subset \mathcal{D}(F), \quad \mathcal{D}(B_b^{1/2}) \subset \mathcal{D}(G), \tag{41}$$

Trans. (40), (41) (S 2.2., from (20) to (23)–(25)) $\Longrightarrow$ oper. matrice $\mathcal{A}_{a,b,g}$ arises:

$$\mathcal{A}_{a,b,g} := \begin{pmatrix} F_a + iG & iB_b^{1/2} \\ iB_b^{1/2} & aI \end{pmatrix}, \ \mathcal{D}(\mathcal{A}_{a,b,g}) = \mathcal{D}(B_b^{1/2}) \oplus \mathcal{D}(B_b^{1/2}), \tag{42}$$

problem (40), (41) trans. into Cauchy Problem

$$\frac{dz}{dt} + \mathcal{A}_{a,b,g} z = f_{a,u}(t), \quad z(0) = y^0 = (u^1; -iB_b^{1/2}u^0)^\tau.$$

Lemma 2.3

$\mathcal{A}_{a,b,g}$, that defined on $\mathcal{D}(\mathcal{A}_{a,b,g})$ by formula (42), is a max. uniformly accr. oper. and inequality (26) is satisfied.

Ideas of proof:

Repeat the proof of Lemma 2.2, using instead $T_{a,b} := (a^{-1}I + B_b^{-1/2}F_aB_b^{-1/2})^{-1}$

$$T_{a,b,g} := (a^{-1}I + B_b^{-1/2}F_aB_b^{-1/2} + iB_b^{-1/2}GB_b^{-1/2})^{-1} \in \mathcal{L}(\mathcal{H}), \qquad (43)$$

$$\|T_{a,b,g}\| \leqslant a.$$

The estimation above follows from

$$\mathrm{Re}((a^{-1}I + B_b^{-1/2}F_aB_b^{-1/2} + iB_b^{-1/2}GB_b^{-1/2})z_1, z_1)_{\mathcal{H}} \geqslant a^{-1}\|z_1\|^2_{\mathcal{H}}.$$

(41) $\implies$ in (43) $B_b^{-1/2}GB_b^{-1/2}$ is self-adjoint and bounded operator.
Instead (32) we have

$$\mathcal{A}^{-1}_{a,b,g} = \begin{pmatrix} B_b^{-1/2}T_{a,b,g}B_b^{-1/2} & -ia^{-1}B_b^{-1/2}T_{a,b,g} \\ -ia^{-1}T_{a,b,g}B_b^{-1/2} & (B_b^{-1/2}F_aB_b^{-1/2} + iB_b^{-1/2}GB_b^{-1/2})T_{a,b,g} \end{pmatrix},$$

instead (33) we have

$$\mathcal{R}(\mathcal{A}^{-1}_{a,b,g}) = \mathcal{D}(\mathcal{A}_{a,b,g}) = \mathcal{D}(B_b^{1/2}) \oplus \mathcal{D}(B_b^{1/2}). \qquad (44)$$

Theorem 2.3

Let conditions (35) be satisfied.
Then problem (39), (38), (41) has a uniq. s.s. on $[0, T]$.

Ideas of proof:
Repeat the proof of Th. 2.2 using the fact that $(-\mathcal{A}_{a,b,g})$ is a gener. of a semigroup $\mathcal{U}(t)$, and estimation (34) holds true, the constant $c > 0$ is the same one.

□

Consider Cauchy problem for integro-diff. eq. (1) using the propos. proved for diff. eq. (2).

The simplest case: cond. (3), (4) are satisfied.

Act to integr.-diff. eq. (1) the same trans., as in S. 2.1 (for (2)) $\Longrightarrow$ Cauchy Pr. arises

$$\frac{dy}{dt} + \mathcal{A}_0 y + \sum_{k=1}^{m} \int_0^t \mathcal{V}_k(t,\xi)\mathcal{A}_k y(\xi)d\xi = \widehat{f}_0(t), \quad y(0) = y^0, \tag{45}$$

$$y(t) := \left(\frac{du}{dt}; \frac{dv}{dt}\right)^{\tau}, \quad y^0 := (u^1; -iB^{1/2}u^0)^{\tau}, \quad \widehat{f}_0(t) := (f(t) - \varphi_0(t); 0)^{\tau},$$

$$\mathcal{A}_0 := i\begin{pmatrix} I & -iFB^{-1/2} \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & B^{1/2} \\ B^{1/2} & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(B^{1/2}) \oplus \mathcal{D}(B^{1/2}),$$

$$\mathcal{A}_k = \operatorname{diag}(A_k; 0), \quad \mathcal{V}_k(t,\xi) = \operatorname{diag}(V_k(t,\xi); 0), \quad k = \overline{1,m}, \tag{46}$$

$$V_k(t,\xi) = \int_\xi^t G_k(t,s)ds, \quad k = \overline{1,m}, \quad \varphi_0(t) = \sum_{k=1}^{m} \int_0^t G_k(t,s)A_k u^0 ds. \tag{47}$$

Theorem 2.4

Let in Cauchy problem (1) the conditions

$$F = F^* \geqslant 0, \quad B = B^* \gg 0, \quad G = 0, \quad \mathcal{D}(B^{1/2}) \subset \mathcal{D}(F) \subset \mathcal{H};$$

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(B^{1/2}), \quad f(t) \in W_p^1([0,T];\mathcal{H}), \quad p > 1;$$

$$G_k(t,s),\ \partial G_k(t,s)/\partial t \in C(\triangle_T; \mathcal{L}(\mathcal{H})),\ \triangle_T := \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\}; \quad (48)$$

$$\mathcal{D}(A_k) \supset \mathcal{D}(B^{1/2}), \quad k = \overline{1,m}, \quad (49)$$

be satisfied.

Then the problem (45)–(47) has a uniq. s.s. $u(t)$ on $[0,T]$.

Ideas of proof: By scheme Th. 1.7.

Let's prove that assumptions of Th. 1.7. are satisfied.

Actually, by Lemma 2.1 $\mathcal{A}_0$ from (46) — max. accr. $\Longrightarrow$ $(-\mathcal{A}_0)$ — a gener. of a contracting semigroup.

$u^0 \in \mathcal{D}(B)$, $u^1 \in \mathcal{D}(B^{1/2})$ $\Longrightarrow$ $y^0 = (u^1; -iB^{1/2}u^0)^\tau \in \mathcal{D}(\mathcal{A}_0)$,

(48), (47) $\Longrightarrow$ $\varphi_0(t) \in C^1([0,T];\mathcal{H})$ $\Longrightarrow$ $\widehat{f}_0(t) \in W_p^1([0,T];\mathcal{H}^2)$, $p > 1$.

(49) $\Longrightarrow$ $\mathcal{D}(\mathcal{A}_k) = \mathcal{D}(A_k) \oplus \mathcal{H} \supset \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(B^{1/2}) \oplus \mathcal{D}(B^{1/2})$, $k = \overline{1,m}$,

(48) $\Longrightarrow$ $\mathcal{V}_k(t,\xi)$, $\partial\mathcal{V}_k(t,\xi)/\partial t \in C(\triangle_T; \mathcal{L}(\mathcal{H}^2))$, $k = \overline{1,m}$.

$\Longrightarrow$ problem (45)–(47) has a uniq. s.s. on $[0,T]$.

As corollary from Th. 2.4 :

Theorem 2.5

Let assumptions of Th. 2.4. be satisfied.
$\Longrightarrow$ Cauchy Problem (1) for second order integro-diff. eq. has a uniq. s.s. on $[0, T]$.

Ideas of proof:
If the cond. of Th. 2.4 are satisfied $\Longrightarrow$ Cauchy Pr. (45) has a uniq. s.s. on $[0, T]$ $\Longrightarrow$

$$\frac{dy_1}{dt} + Fy_1 + iB^{1/2}y_2 + \sum_{k=1}^{m} \int_0^t V_k(t,\xi)A_k y_1(\xi)d\xi = f(t) - \sum_{k=1}^{m} \int_0^t G_k(t,s)A_k u^0 ds, \tag{50}$$

$$\frac{dy_2}{dt} + iB^{1/2}y_1 = 0, \quad y_1(0) = u^1, \quad y_2(0) = -iB^{1/2}u^0, \tag{51}$$

and all items — contin. functions with values in $\mathcal{H}$.
Introduce the functions

$$u(t) := \int_0^t y_1(\xi)d\xi + u^0, \quad v(t) := \int_0^t y_2(\xi)d\xi, \tag{52}$$

except $v(t)$ from (50), (51) $\Longrightarrow$ $u(t)$ — s.s. to (1).

Now we study integro-diff. eq. with the main part that is not parab. or hyperb. (in sense S. 3.1, S. 3.2).

Let's consider Cauchy Problem

$$\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + Bu + \sum_{k=1}^{m} \int_0^t G_k(t,s)A_k u(s)ds = f(t), \quad (53)$$

$$u(0) = u^0, \quad u'(0) = u^1,$$

and the "shortened" problem

$$\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + Bu = f(t), \quad u(0) = u^0, \quad u'(0) = u^1. \quad (54)$$

Assumptions: oper. coeff. are comparable by another way.

Assumptions:

$$G = 0, \quad F \gg 0, \quad B \gg 0, \quad \mathcal{D}(B) \subset \mathcal{D}(F) \subset \mathcal{D}(B^{1/2}). \tag{55}$$

Consider problem (54), (55) has s.s. on $[0, T]$. Introduce $v(t)$ by the formulas

$$\frac{dv}{dt} = -iB^{1/2}u(t), \quad v(0) = 0. \tag{56}$$

s.s. $u(t) \in C^1([0, T]; \mathcal{D}(F)) \implies dv/dt \in C^1([0, T]; \mathcal{D}(F)) \implies$

$$\frac{d^2v}{dt^2} + iB^{1/2}\frac{du}{dt} = 0, \quad \frac{dv}{dt}(0) = -iB^{1/2}u^0.$$

$\implies$ (54), (55) trans. to Cauchy Problem in $\mathcal{H}^2$:

$$\frac{dy}{dt} + \mathcal{A}_0 y = f_0(t), \quad y(0) = y^0, \tag{57}$$

$$y(t) := (du/dt; dv/dt)^\tau, \quad y^0 := (u^1; -iB^{1/2}u^0)^\tau, \quad f_0(t) := (f(t); 0)^\tau,$$

$$\mathcal{A}_0 := \begin{pmatrix} F & iB^{1/2} \\ iB^{1/2} & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_0) := \mathcal{D}(F) \oplus \mathcal{D}(B^{1/2}). \tag{58}$$

$\mathcal{D}(F) \subset \mathcal{D}(B^{1/2}) \implies \mathcal{A}_0$ def. correctly on $\mathcal{D}(\mathcal{A}_0)$.
$\mathcal{A}_0$ is accr. on $\mathcal{D}(\mathcal{A}_0)$:

$$\mathrm{Re}(\mathcal{A}_0 y, y)_{\mathcal{H}^2} \geqslant 0, \quad \forall y \in \mathcal{D}(\mathcal{A}_0). \tag{59}$$

To trans. to Cauchy Problem with uniformly accr. oper. we introduce an unknown function:

$$y(t) = \mathrm{e}^{at} z(t), \quad a > 0, \quad \Longrightarrow$$

$$\frac{dz}{dt} + \mathcal{A}_a z = \mathrm{e}^{-at} f_0(t) =: f_a(t), \quad z(0) = y(0) = y^0, \tag{60}$$

$$\mathcal{A}_a := \mathcal{A}_0 + aJ = \begin{pmatrix} F_a & iB^{1/2} \\ iB^{1/2} & aI \end{pmatrix}, \; F_a := F + aI, \; \mathcal{D}(\mathcal{A}_a) := \mathcal{D}(\mathcal{A}_0). \tag{61}$$

Lemma 3.1

$\mathcal{A}_a$ admits the factorization

$$\mathcal{A}_a = \begin{pmatrix} I & 0 \\ iQ_a & I \end{pmatrix} \begin{pmatrix} F_a & 0 \\ 0 & aI + V_a V_a^+ \end{pmatrix} \begin{pmatrix} I & iQ_a^+ \\ 0 & I \end{pmatrix}, \tag{62}$$

$$Q_a := B^{1/2} F_a^{-1}, \; \mathcal{D}(Q_a) = \mathcal{H}, \; Q_a^+ := F_a^{-1} B^{1/2}, \; \mathcal{D}(Q_a^+) = \mathcal{D}(B^{1/2}), \tag{63}$$

$$V_a := B^{1/2} F_a^{-1/2}, \quad \mathcal{D}(V_a) = \mathcal{R}(V_a^{-1}) = \mathcal{R}(F_a^{1/2} B^{-1/2}), \tag{64}$$

$$V_a^+ := F_a^{-1/2} B^{1/2}, \quad \mathcal{D}(V_a^+) = \mathcal{D}(B^{1/2}). \tag{65}$$

Ideas of proof:

Factorization (62) is verified immediately, taking into account of (63)–(65).

(55) $\Longrightarrow$ $Q_a = (B^{1/2} F^{-1})(I + aF^{-1})^{-1}$ is bounded $\Longrightarrow$ is defined on all $\mathcal{H}$.

(55) $\Longrightarrow$ $V_a^{-1} = F_a^{1/2} B^{-1/2} = (I + aF^{-1})^{1/2} (F^{1/2} B^{-1/2})$ is bounded $\Longrightarrow$ is defined on all $\mathcal{H}$ $\Longrightarrow$ V_a is defined on $\mathcal{R}(V_a^{-1})$ and is unbounded oper (in general).

Lemma 3.2

$$Q_a^+ = Q_a^*|\mathcal{D}(B^{1/2}), \quad \overline{Q_a^+} = Q_a^* \in \mathcal{L}(\mathcal{H}), \tag{66}$$

$$V_a^+ = V_a^*|\mathcal{D}(B^{1/2}),\ \overline{V_a^+} = V_a^* : \mathcal{D}(V_a^*) = \mathcal{R}((V_a^*)^{-1}) \subset \mathcal{H},\ \mathcal{R}(V_a^*) = \mathcal{H}. \tag{67}$$

Ideas of proof: (66) is verified immediately.
(67) is verified immediately (with the distinction $V_a = (V_a^{-1})^{-1}$ is unbounded oper. and therefore $V_a^* = ((V_a^{-1})^*)^{-1}$ is unbounded oper.)

□

As a corollary from Lemmas 3.1, 3.2:

Theorem 3.1

$\mathcal{A}_a$, $\mathcal{D}(\mathcal{A}_a)$ from (61), (58), admits a closing to a max. uniformly accr. oper.

$$\mathcal{A} := \overline{\mathcal{A}_a} = \begin{pmatrix} I & 0 \\ iQ_a & I \end{pmatrix} \begin{pmatrix} F_a & 0 \\ 0 & aI + V_a V_a^* \end{pmatrix} \begin{pmatrix} I & iQ_a^* \\ 0 & I \end{pmatrix} \tag{68}$$

$$\mathcal{D}(\mathcal{A}) := \{u = (u_1; u_2)^\tau : u_1 \in \mathcal{D}(B^{1/2}), \quad u_1 + iQ_a^* u_2 \in \mathcal{D}(F_a)\}, \tag{69}$$

$$\mathcal{A}u = \begin{pmatrix} F_a(u_1 + iQ_a^* u_2) \\ iB^{1/2}u_1 + au_2 \end{pmatrix}, \quad u \in \mathcal{D}(\mathcal{A}). \tag{70}$$

Ideas of proof:

Lemmas 3.1, 3.2: Replace Q_a^+, V_a^+ with Q_a^*, V_a^* in (62) $\Longrightarrow$ $\mathcal{A}_a$ — from $\mathcal{A}$ $\Longrightarrow$ (68).
$\mathcal{A}_0$ is accr. $\Longrightarrow$ $\mathcal{A}_a = \mathcal{A}_0 + a\mathcal{I}$ is uniformly accr.:

$$\mathrm{Re}(\mathcal{A}_a y, y)_{\mathcal{H}^2} \geqslant a\|y\|^2_{\mathcal{H}^2}, \quad y \in \mathcal{D}(\mathcal{A}_a) = \mathcal{D}(\mathcal{A}_0) \quad \Longrightarrow$$

After closing $\mathcal{A}$ is uniformly accr.: $\mathrm{Re}(\mathcal{A}y, y)_{\mathcal{H}^2} \geqslant a\|y\|^2_{\mathcal{H}^2}, \quad a > 0, \quad y \in \mathcal{D}(\mathcal{A})$.
Above and (68) $\Longrightarrow$ $\mathcal{A}$ — is max. uniformly accr. and has bounded inverse oper.:

$$\mathcal{A}^{-1} = \begin{pmatrix} I & -iQ_a^* \\ 0 & I \end{pmatrix} \begin{pmatrix} F_a^{-1} & 0 \\ 0 & (aI + V_aV_a^*)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -iQ_a & I \end{pmatrix}, \qquad \|\mathcal{A}^{-1}\| \leqslant a^{-1}.$$

Let $u = (u_1; u_2)^\tau \in \mathcal{D}(\mathcal{A})$. From (68) $\Longrightarrow$

$$\mathcal{A}u = \begin{pmatrix} F_a(u_1 + iQ_a^* u_2) \\ iQ_aF_a(u_1 + iQ_a^*u_2) + (aI + V_aV_a^*)u_2 \end{pmatrix} \quad \Longrightarrow \qquad (71)$$

$$u_1 + iQ_a^*u_2 \in \mathcal{D}(F_a), \quad u_2 \in \mathcal{D}(V_aV_a^*).$$

Let $u_2 \in \mathcal{D}(B^{1/2})$. By Lemma 3.2 and (55) $\Longrightarrow$

$$Q_a^*u_2 = Q_a^+u_2 = F_a^{-1}B^{1/2}u_2 \in \mathcal{D}(F_a) = \mathcal{D}(F) \subset \mathcal{D}(B^{1/2}).$$

Take into account $Q_aF_a = B^{1/2}$: from (71) $\Longrightarrow$ $u_1 \in \mathcal{D}(B^{1/2})$ and

$$iB^{1/2}u_1 - B^{1/2}F_a^{-1}B^{1/2}u_2 + au_2 + V_aV_a^*u_2 = iB^{1/2}u_1 + au_2, \; u_2 \in \mathcal{D}(B^{1/2}). \quad (72)$$

$\mathcal{D}(B^{1/2})$ is dense in $\mathcal{H}$ $\Longrightarrow$ Close on $u_2 \in \mathcal{D}(B^{1/2})$ $\Longrightarrow$ (72), $\forall u_2 \in \mathcal{H}$ $\Longrightarrow$ (69), (70).

Consider the problem that is "associated" with (54), (55):

$$\frac{d^2u}{dt^2} + F\Big(\frac{du}{dt} + Q_a^* B^{1/2} u\Big) + aQ_a^* B^{1/2} u = f(t), \quad u(0) = u^0, \quad u'(0) = u^1, \qquad (73)$$

$$Q_a^* = (B^{1/2} F_a^{-1})^* = \overline{Q_a^+}, \quad F_a := F + aI, \quad a > 0.$$

Definition 3.1

A function $u(t)$ with values in $\mathcal{H}$ is called s.s. to "associated" pr. (73) on $[0, T]$ if:
1°. $u(t) \in \mathcal{D}(B^{1/2})$, $t \in [0, T]$ and $B^{1/2}u(t) \in C([0, T]; \mathcal{H})$;
2°. $du/dt + Q_a^* B^{1/2} u \in \mathcal{D}(F)$ and $F(du/dt + Q_a^* B^{1/2} u) \in C([0, T]; \mathcal{H})$;
3°. $u(t) \in C^2([0, T]; \mathcal{H})$;
4°. $\forall t \in [0, T]$ eq. (73) holds true; 5°. initial cond. (73) are satisfied.

Lemma 3.3

If s.s. $u(t)$ of "associated" pr. (73) is satisfied with additional smoothness cond.

$$u(t) \in \mathcal{D}(B), \quad Bu(t) \in C([0, T]; \mathcal{H}), \quad \Longrightarrow \qquad (74)$$

$u(t)$ is a s.s. to problem (54),(55).

Ideas of proof:
(74), (66) $\Longrightarrow$ $Q_a^* B^{1/2} u(t) = Q_a^+ B^{1/2} u(t) = F_a^{-1} B u(t) \in C([0, T]; \mathcal{D}(F))$.
(73) $\Longrightarrow$ $d^2u/dt^2 + F du/dt + (F_a - aI) F_a^{-1} B u(t) + a F_a^{-1} B u(t) = f(t)$.
$G = 0$ $\Longrightarrow$ (54).

Theorem 3.2

Let the cond.

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(F), \quad f(t) \in W_p^1([0,T];\mathcal{H}), \quad p > 1, \tag{75}$$

and (55) be satisfied. $\Longrightarrow$ $\exists$ a uniq. s.s. to "associated" problem (73) on $[0,T]$.
If additional smoothness conditions (74) are satisfied $\Longrightarrow$ $\exists$ a uniq. s.s. to problem (54), (55) for $t \in [0,T]$.

Ideas of proof:
By Lemma 3.3 it is enough to prove the first propos. of Th. 3.2.
Consider instead (60) Cauchy Problem

$$\frac{dz}{dt} = -\mathcal{A}z + f_a(t), \quad z(0) = y^0, \tag{76}$$

with $\mathcal{A}$ from (68), (70).
If (75) $\Longrightarrow$ in (76)

$$y^0 = (u^1; -iB^{1/2}u^0)^\tau \in \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A}_a) \subset \mathcal{D}(\mathcal{A}), \tag{77}$$

$$f_a(t) = \mathrm{e}^{-at}(f(t); 0)^\tau \in W_p^1([0,T];\mathcal{H}^2), \quad p > 1. \tag{78}$$

$\Longrightarrow$ by Th. 3.1 $(-\mathcal{A})$ is a max. uniformly diss. oper. $\Longrightarrow$ it is a gener. of a contracting (C_0)–semigroup $\mathcal{U}(t)$ and

$$\|\mathcal{U}(t)\| \leqslant \mathrm{e}^{-at}, \quad a > 0, \quad t \geqslant 0.$$

By Th. of S.Yakubov $\Longrightarrow$ problem (76) has a uniq. s.s. on $[0,T]$ $\Longrightarrow$

$$\frac{dz_1}{dt} + F_a(z_1 + iQ_a^* z_2) = \mathrm{e}^{-at} f(t), \quad z_1(0) = u^1, \tag{79}$$

$$\frac{dz_2}{dt} + iB^{1/2} z_1 + a z_2 = 0, \quad z_2(0) = -iB^{1/2} u^0, \tag{80}$$

all items — contin. functions on t with values $\mathcal{H}$, $z_1(t) + iQ_a^* z_2(t) \in C([0,T]; \mathcal{D}(F_a))$.
From (80) $\Longrightarrow$

$$z_2(t) = -i\mathrm{e}^{-at} B^{1/2} u^0 - i \int_0^t \mathrm{e}^{-a(t-s)} B^{1/2} z_1(s) ds.$$

Subst. this expression in (79) $\Longrightarrow$

$$\frac{dz_1}{dt} + F_a\Big(z_1(t) + \mathrm{e}^{-at} Q_a^* B^{1/2} u^0 + Q_a^* \int_0^t \mathrm{e}^{-a(t-s)} B^{1/2} z_1(s) ds\Big) = f(t)\mathrm{e}^{-at}, \quad z_1(0) = u^1. \tag{81}$$

$u^0 \in \mathcal{D}(B)$ $\Longrightarrow$ $Q_a^* B^{1/2} u^0 = Q_a^+ B^{1/2} u^0 = F_a^{-1} B u^0 \in \mathcal{D}(F_a)$, and (81) $\Longrightarrow$

$$\frac{dz_1}{dt} + (F + aI)\Big(z_1(t) + \int_0^t \mathrm{e}^{-a(t-s)} Q_a^* B^{1/2} z_1(s) ds\Big) = f(t)\mathrm{e}^{-at} - \mathrm{e}^{-at} B u^0, \quad z_1(0) = u^1.$$

The inverse substitution $z_1(t)$: $z_1(t) = \mathrm{e}^{-at} y_1(t) = \mathrm{e}^{-at} du/dt.$ $\Longrightarrow$

$$e^{-at}\frac{d^2u}{dt^2} - ae^{-at}\frac{du}{dt} + (F + aI)\Big(e^{-at}\frac{du}{dt} + e^{-at}\int_0^t Q_a^* B^{1/2} e^{as} e^{-as}\frac{du}{ds}ds\Big) =$$

$$= f(t)e^{-at} - e^{-at}Bu^0, \quad u(0) = u^0, \quad u'(0) = u^1. \quad (82)$$

In (80) $B^{1/2}z_1(t) = e^{-at}B^{1/2}(du/dt)$ — contin. on $t \in [0,T]$, Q_a^* — bounded $\Longrightarrow$

$$\int_0^t Q_a^* B^{1/2}\frac{du(s)}{ds}ds = Q_a^* B^{1/2}(u(t) - u^0).$$

From above and (82) $\Longrightarrow$ (73) are satisfied, and all items in (73) — contin. on $t \in [0,T]$.

□

Let's weaken the cond. on $f(t)$ in (54), amplifying cond. (55) on the domains of oper. coeff.

Definition 3.2

B is called A-compact, if A has bounded inverse oper. A^{-1} and $BA^{-1} \in \mathfrak{S}_\infty(\mathcal{H})$.

Theorem 3.3

Let cond. (55) be satisfied and let $B^{1/2}$ be F-compact.
If the cond.

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(F), \quad f(t) \in C^\gamma([0,T];\mathcal{H}), \quad 0 < \gamma \leqslant 1, \tag{83}$$

are satisfied, then "associated" Cauchy Problem (73) has a uniq. s.s. on $[0,T]$.

Ideas of proof:
If $B^{1/2}$ is F–compact $\implies$

$$Q_a = B^{1/2}F_a^{-1} = (B^{1/2}F^{-1})FF_a^{-1} = (B^{1/2}F^{-1})(I + aF^{-1})^{-1} \in \mathfrak{S}_\infty(\mathcal{H}).$$

$\implies$ in (68) operator

$$\mathcal{A} = (\mathcal{I} + \mathcal{S}_1)\mathcal{A}_{00}(\mathcal{I} + \mathcal{S}_2), \quad \mathcal{S}_j \in \mathfrak{S}_\infty(\mathcal{H}^2), \quad j = 1, 2, \tag{84}$$

is a weakly perturbation of self-adjont unbounded positive definite operator

$$\mathcal{A}_{00} := \operatorname{diag}(F_a; aI + V_aV_a^*), \quad D(\mathcal{A}_{00}) := D(F_a) \oplus D(V_aV_a^*), \tag{85}$$

and left and right factors are invertible and inverse operators have the same structure.

Repeat the proof of Th. 3.2 for Cauchy Pr. (76). Here cond. (77) are satisfied, too.

Replace in (76) with $\mathcal{A}$ from (84) by the formula:

$$(\mathcal{I} + \mathcal{S}_2)z(t) =: w(t). \tag{86}$$

Act to the both parts of obtained eq. by bounded and boundedly invert. oper. $(\mathcal{I}+\mathcal{S}_2)$ $\Longrightarrow$ equivalent Cauchy problem

$$\frac{dw}{dt} = -(\mathcal{I}+\mathcal{S}_2)(\mathcal{I}+\mathcal{S}_1)\mathcal{A}_{00}w + f_a(t), \quad w(0) = (u^1 + Q_a^* B^{1/2}u^0; -iB^{1/2}u^0)^\tau, \quad (87)$$

$$(\mathcal{I}+\mathcal{S}_2)f_a(t) \equiv f_a(t).$$

For self-adjoint positive defined oper. $\mathcal{A}_{00}$ from (85) by Lemma 3.2 $\Longrightarrow$

$$\mathcal{D}(\mathcal{A}_{00}) \supset \mathcal{D}(\mathrm{diag}(F_a; aI + V_aV_a^+)) = \mathcal{D}(F_a) \oplus \mathcal{D}(B^{1/2}) = \mathcal{D}(\mathcal{A}_0).$$

(84) $\Longrightarrow$ $(\mathcal{I}+\mathcal{S}_2)(\mathcal{I}+\mathcal{S}_1) =: (\mathcal{I}+\mathcal{S})$, $\mathcal{S} \in \mathfrak{S}_\infty(\mathcal{H}^2)$, $\Longrightarrow$ (87) is a parab. eq. $(-(\mathcal{I}+\mathcal{S})\mathcal{A}_{00})$ is a gener. of a semigroup that analyt. in a sector $\supset (t>0)$.

(83) $\Longrightarrow$ in (87) $f_a(t) \in C^\gamma([0,T];\mathcal{H}^2)$.

In (87) $w(0) \in \mathcal{D}(\mathcal{A}_{00})$. Actually, if $u^0 \in \mathcal{D}(B)$, $u^1 \in \mathcal{D}(F)$ $\Longrightarrow$ $-iB^{1/2}u^0 \in \mathcal{D}(B^{1/2})$, and $u^1 + Q_a^* B^{1/2}u^0 = u^1 + Q_a^+ B^{1/2}u^0 = u^1 + F_a^{-1}Bu^0 \in \mathcal{D}(F)$.

$$\mathcal{D}(V_aV_a^*) \supset \mathcal{D}(V_aV_a^+) = \mathcal{D}(B^{1/2}F_a^{-1}B^{1/2}) = \mathcal{D}(Q_aB^{1/2}) = \mathcal{D}(B^{1/2}) \Longrightarrow w(0) \in \mathcal{D}(\mathcal{A}_{00}).$$

So, $w(0) \in \mathcal{D}(\mathcal{A}_{00})$, $f_a(t) \in C^\gamma([0,T];\mathcal{H}^2)$, $\Longrightarrow$ (87) has a uniq. s.s. on $[0,T]$. (Goldstein)

Using inverse subst. (86) return to problem (76) $\Longrightarrow$ (76) has a s.s. on $[0,T]$.

Repeat the steps of Th. 3.2, from eq. (79), (80) ...

Remark 3.1

In problem (54), (55), above, we assumed that gyrosc. oper $G = 0$.

However, if $0 \neq G = G^* \in \mathcal{L}(\mathcal{H})$, then all the propos. of Th. 3.1–3.3 hold true.

In this case in right part of (68) the item $i\,\mathit{diag}(G; 0)$ is added,
and in (70) we take

$$\mathcal{A}u = \begin{pmatrix} F_a(u_1 + iQ_a^* u_2) + iGu_1 \\ iB^{1/2}u_1 + au_2 \end{pmatrix},$$

instead (73) we consider the Cauchy problem

$$\frac{d^2u}{dt^2} + F\Big(\frac{du}{dt} + Q_a^* B^{1/2} u\Big) + iG\frac{du}{dt} + aQ_a^* B^{1/2} u = f(t),\ u(0) = u^0,\ u'(0) = u^1.$$

3.3. The diss. oper. and poten. energy oper. are bounded from below, gyrosc. oper. is unbounded

Let in (54) : $\quad F \geqslant \gamma_F I, \quad F_a := F + aI \geqslant \alpha_F I, \quad \alpha_F := \gamma_F + a > 0,$ (88)

$$B \geqslant \gamma_B I, \quad B_b := B + bI \geqslant \alpha_B I, \quad \alpha_B := \gamma_B + b > 0,$$

$$\mathcal{D}(B) = \mathcal{D}(B_b) \subset \mathcal{D}(F) = \mathcal{D}(F_a) \subset \mathcal{D}(B_b^{1/2}) \subset \mathcal{D}(F_a^{1/2}), \tag{89}$$

$$G = G^*, \quad \mathcal{D}(G) \supset \mathcal{D}(F_a^{1/2}). \tag{90}$$

The right cond. in (89) is a corollary of left and right Heintz inequalities.

We use another factorization of oper. matrice (61). Use the same trans. as in S. 1.3 $\Longrightarrow$

$$\frac{dz}{dt} + \mathcal{A}_{a,b} z = f_{a,u}(t) := \mathrm{e}^{-at} f_{0,u}(t), \quad z(0) = y^0, \tag{91}$$

$$f_{0,u}(t) = (f(t) + bu(t); 0)^\tau, \quad y^0 = (u^1; -iB_b^{1/2} u^0)^\tau,$$

$$\mathcal{A}_{a,b} := \begin{pmatrix} F_a + iG & iB_b^{1/2} \\ iB_b^{1/2} & aI \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_{a,b}) = \mathcal{D}(F_a) \oplus \mathcal{D}(B_b^{1/2}). \tag{92}$$

$\mathcal{A}_{a,b}$ is a unif. accr. : $\mathrm{Re}(\mathcal{A}_{a,b} z, z)_{\mathcal{H}^2} \geqslant c\|z\|^2_{\mathcal{H}^2}, \ c := \min(\alpha_F; a) > 0, \ z \in \mathcal{D}(\mathcal{A}_{a,b}).$ (93)

Introduce auxil. oper.: $V_{a,b} := B_b^{1/2} F_a^{-1/2}, \ \mathcal{D}(V_{a,b}) = \mathcal{R}(V_{a,b}^{-1}) = \mathcal{R}(F_a^{1/2} B_b^{-1/2}),$

$$\mathcal{R}(V_{a,b}) = \mathcal{D}(F_a^{1/2} B_b^{-1/2}) = \mathcal{H}, \tag{94}$$

$$V_{a,b}^+ := F_a^{-1/2} B_b^{1/2}, \ \mathcal{D}(V_{a,b}^+) = \mathcal{R}(B_b^{-1/2} F_a^{1/2}) \subset \mathcal{D}(B^{1/2}), \ \overline{V_{a,b}^+} = V_{a,b}^*. \tag{95}$$

As in Lemmas 3.1, 3.2 we proved the following facts.

Lemma 3.4

Oper. matrice $\mathcal{A}_{a,b}$ admits the factorization

$$\mathcal{A}_{a,b} = \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I + iG_a & iV_{a,b}^{+} \\ iV_{a,b} & aI \end{pmatrix} \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix}, G_a := F_a^{-1/2} G F_a^{-1/2} \in \mathcal{L}(\mathcal{H}),$$

here $V_{a,b}$, $V_{a,b}^{+}$ — from (94), (95),

$$V_{a,b}^{+} = V_{a,b}^{*}|\mathcal{D}(B_b^{1/2}), \ \overline{V_{a,b}^{+}} = V_{a,b}^{*} : \mathcal{D}(V_{a,b}^{*}) = \mathcal{R}((V_{a,b}^{*})^{-1}) \subset \mathcal{H} \to \mathcal{H}, \ \mathcal{R}(V_{a,b}^{*}) = \mathcal{H}. \quad (96)$$

Theorem 3.4 (the generalization of Th. 3.1)

Oper. matrice $\mathcal{A}_{a,b}$ (from (92)) admits the closing to a max. uniformly accr. oper.

$$\mathcal{A} = \overline{\mathcal{A}_{a,b}} = \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I + iG_a & iV_{a,b}^{*} \\ iV_{a,b} & aI \end{pmatrix} \begin{pmatrix} F_a^{1/2} & 0 \\ 0 & I \end{pmatrix} \quad (97)$$

$$\mathcal{D}(\mathcal{A}) := \{z = (z_1; z_2)^{\tau} : z_1 \in \mathcal{D}(B_b^{1/2}), \quad (I + iG_a)F_a^{1/2} z_1 + iV_{a,b}^{*} z_2 \in \mathcal{D}(F_a^{1/2})\}, \quad (98)$$

$$\mathcal{A}z = \begin{pmatrix} F_a^{1/2}((I + iG_a)F_a^{1/2} z_1 + iV_{a,b}^{*} z_2) \\ iB_b^{1/2} z_1 + a z_2 \end{pmatrix}, \quad z \in \mathcal{D}(\mathcal{A}) \subset \mathcal{H}^2. \quad (99)$$

Ideas of proof: By scheme of proof of Th. 3.1.

Lemma 3.4 $\Longrightarrow$ $\mathcal{A}$ — closing of $\mathcal{A}_{a,b}$ by replace $V_{a,b}^{+}$ to $V_{a,b}^{*}$ $\Longrightarrow$ (97).

(90) $\Longrightarrow$ $G_a = G_a^* \in \mathcal{L}(\mathcal{H})$.

$\mathcal{A}_{a,b}$ — a uniformly accr. oper. $\Longrightarrow$ after the closing $\mathcal{A} = \overline{\mathcal{A}_{a,b}}$ — a uniformly accr. oper. with the same constant $c > 0$.

Note that (98), (99) follow immediately from (97) and def. of $V_{a,b}$.

$\square$

Consider the Cauchy problem

$$\frac{dz}{dt} + \mathcal{A}z = f_{a,u}(t), \quad z(0) = y^0, \tag{100}$$

under cond. (88)–(90). Here $\mathcal{A}$ — a max. uniformly accr. oper.
Taking into account of

$$u(t) = u^0 + \int_0^t u'(\xi)d\xi = u^0 + \int_0^t y_1(\xi)d\xi = u^0 + \int_0^t \mathrm{e}^{a\xi} z_1(\xi)d\xi,$$

$z(t) = \mathrm{e}^{-at}y(t) = \mathrm{e}^{-at}(u'(t); v'(t))^\tau$
(100) trans. to the form

$$\frac{dz}{dt} + \mathcal{A}z - b\Big(\int_0^t \mathrm{e}^{-a(t-\xi)} z_1(\xi)d\xi; 0\Big)^\tau = \mathrm{e}^{-at}(f(t) + bu^0; 0)^\tau, \quad z(0) = y^0. \tag{101}$$

Definition 3.3

The Cauchy Problem

$$\frac{d^2u}{dt^2} + iG\frac{du}{dt} + F_a^{1/2}\Big(F_a^{1/2}\frac{du}{dt} + V_{a,b}^* B_b^{1/2}u\Big) - \\ - a\frac{du}{dt} - bu = f(t), \quad u(0) = u^0, \quad u'(0) = u^1, \tag{102}$$

is called the "associated" with Cauchy Problem (54), (88)–(90).

Lemma 3.5

If s.s. $u(t)$ to problem (102) is satisfied with additional smoothness conditions

$$u(t) \in \mathcal{D}(B), \quad Bu(t) \in C([0,T];\mathcal{H}), \tag{103}$$

$\Longrightarrow \quad u(t)$ is a s.s. to problem (54), (88)–(90).

Ideas of proof:

Let (103) be satisfied.

(96) $\quad \Longrightarrow$

$$B_b u(t) \in C([0,T];\mathcal{H}), \quad B_b^{1/2} u(t) \in C([0,T];\mathcal{D}(B_b^{1/2})),$$
$$V_{a,b}^* B_b^{1/2} u(t) = V_{a,b}^+ B_b^{1/2} u(t) = F_a^{-1/2} B_b u(t) \in C([0,T];\mathcal{D}(F_a^{1/2})) \quad \Longrightarrow$$

eq. (102) trans. into eq.(54), and all items — contin. on $t \in [0,T]$.

□

Theorem 3.5

Let in problem (54), (88)–(90) the conditions

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(F), \quad f(t) \in W_p^1([0,T];\mathcal{H}), \quad p > 1, \tag{104}$$

be satisfied. Then Cauchy Pr. (100) and "associated" pr. (102) have s.s. on $[0,T]$.

Ideas of proof: Under cond. (104) for problem (101) the cond.

$$y^0 = (u^1; -iB_b^{1/2}u^0)^\tau \in \mathcal{D}(F_a) \oplus \mathcal{D}(B_b^{1/2}) = \mathcal{D}(\mathcal{A}_{a,b}) \subset \mathcal{D}(\mathcal{A}),$$

$$\mathrm{e}^{-at}(f(t) + bu^0; 0)^\tau \in W_p^1([0,T];\mathcal{H}), \quad p > 1,$$

are satisfied.
As in steps 1)–3) of proof of Th. 1.3 $\Longrightarrow$ problem (101) has a s.s. on $[0,T]$.
For this problem the eq. and initial cond. are satisfied:

$$\frac{dz_1}{dt} + F_a^{1/2}[(I + iG_a)F_a^{1/2}z_1 + iV_{a,b}^* z_2] - \\ - b\int_0^t \mathrm{e}^{-a(t-\xi)}z_1(\xi)d\xi = \mathrm{e}^{-at}(f(t) + bu^0), \quad z_1(0) = u^1, \quad (105)$$

$$\frac{dz_2}{dt} + az_2 + iB_b^{1/2}z_1 = 0, \quad z_2(0) = -iB_b^{1/2}u^0,$$

and all items in eq. — contin. on $t \in [0,T]$. $\Longrightarrow$
$z_1(t) =: du/dt \Longrightarrow \mathrm{e}^{at}z_2(t) = -iB_b^{1/2}u(t)$. Subst. in (105) $\Longrightarrow$ (102).
Note that for the solutions of (101) from the property $z(t) \in C([0,T];\mathcal{D}(\mathcal{A})) \Longrightarrow$
$z_1(t) \in C([0,T];\mathcal{D}(B_b^{1/2})) \subset C([0,T];\mathcal{D}(F_a^{1/2})) \subset C([0,T];\mathcal{D}(G)) \Longrightarrow$
$iG_aF_a^{1/2}z_1(t) = iF_a^{-1/2}Gz_1(t) \in C([0,T];\mathcal{D}(F_a^{1/2}))$.

□

Consider problem (53) for integro-diff. eq.:

$$\frac{d^2u}{dt^2} + (F + iG)\frac{du}{dt} + Bu + \\ + \sum_{k=1}^{m} \int_0^t G_k(t,s) A_k u(s) ds = f(t), \quad u(0) = u^0, \quad u'(0) = u^1. \quad (106)$$

The simplest case:

$$F \gg 0, \quad B \gg 0, \quad G = 0, \quad \mathcal{D}(B) \subset \mathcal{D}(F) \subset \mathcal{D}(B^{1/2}) \subset \mathcal{D}(F^{1/2}). \quad (107)$$

The trans. in eq. (106) for "associated" pr. (54) $\Longrightarrow$ the generalization of pr. (60):

$$\frac{dz}{dt} + \mathcal{A}_a z + \Big(\sum_{k=1}^{m} \int_0^t \mathrm{e}^{-a(t-\xi)} V_k(t,\xi) A_k z_1(\xi) d\xi; 0 \Big)^\tau = \\ = \mathrm{e}^{-at} (f(t); 0)^\tau, \quad z(0) = (u^1; -iB^{1/2}u^0)^\tau, \quad (108)$$

then a corresponding problem with $\mathcal{A} := \overline{\mathcal{A}_a}$ arises:

$$\frac{dz}{dt} + \mathcal{A}z + \Big(\sum_{k=1}^{m} \int_0^t \mathrm{e}^{-a(t-\xi)} V_k(t,\xi) A_k z_1(\xi) d\xi; 0\Big)^\tau =$$

$$= \mathrm{e}^{-at}(f(t);0)^\tau, \quad z(0) = (u^1; -iB^{1/2}u^0)^\tau, \quad (109)$$

$$V_k(t,\xi) = \int_\xi^t G_k(t,s) ds, \quad z(t) = \mathrm{e}^{-at} y(t) = \mathrm{e}^{-at}(du/dt; -iB^{1/2}u(t))^\tau.$$

Here:
oper. matrice $\mathcal{A}_a$ def. by (61), (62) on the domain $\mathcal{D}(\mathcal{A}_a) = \mathcal{D}(F) \oplus \mathcal{D}(B^{1/2})$,
oper. matrice $\mathcal{A}$ def. by (68) on the domain (69).

Definition 3.4

The problem

$$\frac{d^2u}{dt^2} + F\Big(\frac{du}{dt} + Q_a^* B^{1/2} u\Big) + aQ_a^* B^{1/2} u +$$

$$+ \sum_{k=1}^{m} \int_0^t G_k(t,s) A_k u(s) ds = f(t), \quad u(0) = u^0, \quad u'(0) = u^1, \quad (110)$$

is called the Cauchy Problem for the eq. that is "associated" with (106).

If s.s. $u(t)$ of "associated" Cauchy Pr. (110) is satisfied with additional smoothness cond.:

$$u(t) \in \mathcal{D}(B), \quad Bu(t) \in C([0,T];\mathcal{H}), \tag{111}$$

$\Longrightarrow$ $u(t)$ is a s.s. of Cauchy Problem (106).
Actually, the proof of Lemma 3.3. isn't connected with integr. items in (110).

Theorem 3.6

Let cond. (107) and the cond.

$$u^0 \in \mathcal{D}(B), \quad u^1 \in \mathcal{D}(F), \quad f(t) \in W_p^1([0,T];\mathcal{H}), \quad p > 1, \tag{112}$$

$$G_k(t,s),\ \partial G_k(t,s)/\partial t \in C(\triangle_T;\mathcal{L}(\mathcal{H})),\ \triangle_T := \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\},\ k = \overline{1,m}, \tag{113}$$

$$\mathcal{D}(A_k) \supset \mathcal{D}(B^{1/2}), \quad k = \overline{1,m}. \tag{114}$$

be satisfied. Then "associated" Cauchy Problem (110) has a uniq. s.s. on $[0,T]$.

Ideas of proof: If cond. (112) $\Longrightarrow$ in problem (109):

$$z(0) = (u^1; -iB^{1/2}u^0)^\tau \in \mathcal{D}(F) \oplus \mathcal{D}(B^{1/2}) \subset \mathcal{D}(\mathcal{A}_a) \subset \mathcal{D}(\mathcal{A}), \tag{115}$$

$$\mathrm{e}^{-at}(f(t);0)^\tau \in W_p^1([0,T];\mathcal{H}^2).$$

$\Longrightarrow$ under cond. (113), (114) problem (109) has a uniq. s.s. $z(t)$ on $[0,T]$. The proof is analog. Th. 3.7.

Note:

$\varphi_0(t) := \sum\limits_{k=1}^{m} \int\limits_0^t G_k(t,s)A_k u(s)ds \in C^1([0,T];\mathcal{H})$, $\mathcal{A}_k := \mathrm{diag}(A_k;0)$ has the property

$$\mathcal{D}(\mathcal{A}_k) = D(A_k) \oplus \mathcal{H} \supset \mathcal{D}(B^{1/2}) \oplus \mathcal{H} \supset \mathcal{D}(\mathcal{A}) =$$
$$= \{(z_1; z_2)^\tau : z_1 \in \mathcal{D}(B^{1/2}),\ z_1 + iQ_a^* z_2 \in \mathcal{D}(F_a)\}.$$

From the existence of s.s. to problem (109) on $[0,T]$ $\Longrightarrow$

$$\frac{dz_1}{dt} + F_a(z_1 + iQ_a^* z_2) + \sum_{k=1}^{m} \int\limits_0^t \mathrm{e}^{-a(t-\xi)} V_k(t,\xi) A_k z_1(\xi) d\xi = \mathrm{e}^{-at} f(t),\ z_1(0) = u^1, \quad (116)$$

$$\frac{dz_2}{dt} + iB^{1/2} z_1 + a z_2 = 0, \quad z_2(0) = -iB^{1/2} u^0, \quad (117)$$

and in eq. all items — contin. on $t \in [0,T]$.
Subst. $z_1(t) = \mathrm{e}^{-at} du/dt$, $z_2(t) = \mathrm{e}^{-at}(-iB^{1/2}u)$
Inverse trans. $\Longrightarrow$ $u(t)$ is a s.s. to "associated" problem (110).

□

Theorem 3.7

Let in problem (110):
a) cond. (107), (112)–(114) be satisfied,
b) cond. on $f(t)$ in (112) replace by the cond. $\quad f(t) \in C^{\gamma}([0,T];\mathcal{H}), \quad 0 < \gamma \leqslant 1,$
c) оператор $B^{1/2}$ is F–compact $\quad (B^{1/2}F^{-1} \in \mathfrak{S}_\infty(\mathcal{H}))$.
$\Longrightarrow$ Cauchy Pr. (110) has a s.s. on $[0,T]$.

Ideas of proof: In the proof of Th. 3.7 we use Sh.–Fr. factoriz. for $\mathcal{A}$ and (68)–(70). Namely:
$Q_a = B^{1/2}F_a^{-1} = B^{1/2}F^{-1}(I + aF^{-1})^{-1} \in \mathfrak{S}_\infty(\mathcal{H}) \quad \Longrightarrow \quad Q_a^* \in \mathfrak{S}_\infty(\mathcal{H}) \quad \Longrightarrow$
$\mathcal{A}$ is written in the form

$$\mathcal{A} = (\mathcal{I} + \mathcal{S}_1)\mathcal{A}_{00}(\mathcal{I} + \mathcal{S}_2), \quad \mathcal{A}_{00} := \operatorname{diag}(F_a; aI + V_a V_a^*),$$

$$\mathcal{I} + \mathcal{S}_1 := \begin{pmatrix} I & 0 \\ iQ_a & I \end{pmatrix}, \quad \mathcal{I} + \mathcal{S}_2 := \begin{pmatrix} I & iQ_a^* \\ 0 & I \end{pmatrix}, \quad \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{S}_\infty(\mathcal{H}^2),$$

— bounded and boundedly invertible oper. with the same structure:

$$(\mathcal{I} + \mathcal{S}_1)^{-1} = \begin{pmatrix} I & 0 \\ -iQ_a & I \end{pmatrix}, \quad (\mathcal{I} + \mathcal{S}_2)^{-1} = \begin{pmatrix} I & -iQ_a^* \\ 0 & I \end{pmatrix}.$$

The domains of $\mathcal{A}$, $\mathcal{A}_{00}$ are connected:

$$(\mathcal{I} + \mathcal{S}_2)\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_{00}) = \mathcal{D}(F_a) \oplus \mathcal{D}(V_a V_a^*). \qquad (118)$$

Taking into account this facts, in eq. (109) subst. $(\mathcal{I}+\mathcal{S}_2)z(t) =: w(t)$
Act from left by $(\mathcal{I}+S_2) \implies$ the Cauchy Problem arises:

$$\frac{dw}{dt} = -(\mathcal{I}+\mathcal{S})\mathcal{A}_{00}w + (\mathcal{I}+\mathcal{S}_2)\Big(\sum_{k=1}^{m}\int_0^t \mathrm{e}^{-a(t-\xi)}V_k(t,\xi)A_k(w_1 - iQ_a^* w_2)(\xi)d\xi; 0\Big)^{\tau} =$$

$$= (\mathrm{e}^{-at}f(t);0)^{\tau}, \quad w(0) = (u^1 + Q_a^* B^{1/2}u^0; -iB^{1/2}u^0)^{\tau}, \quad (119)$$

$$\mathcal{I}+\mathcal{S} := (\mathcal{I}+\mathcal{S}_2)(\mathcal{I}+S_1), \quad \mathcal{S} \in \mathfrak{S}_{\infty}(\mathcal{H}^2).$$

(taking into account $(\mathcal{I}+\mathcal{S}_2)(\mathrm{e}^{-at}f(t);0)^{\tau} = (\mathrm{e}^{-at}f(t);0)^{\tau}$.)
$z(0) = (u^1; -iB^{1/2}u^0)^{\tau} \in \mathcal{D}(\mathcal{A})$ and (118) $\implies$

$$w(0) = (u^1 + Q_a^* B^{1/2}u^0; -iB^{1/2}u^0)^{\tau} = (u^1 + F_a^{-1}Bu^0; -iB^{1/2}u^0)^{\tau} \in \mathcal{D}(\mathcal{A}_{00}).$$

Besides

$$(\mathrm{e}^{-at}f(t);0)^{\tau} \in C^{\gamma}([0,T];\mathcal{H}^2), \quad 0 < \gamma \leqslant 1.$$

Note:
1°. $-(\mathcal{I}+\mathcal{S})\mathcal{A}_{00}$ — a gener. of a semigroup that analyt. in a sector $\supset (t \geqslant 0)$.
2°. from (115) $\implies$ $\mathcal{A}_k := \mathrm{diag}(A_k;0)$ has the property $\mathcal{D}(\mathcal{A}_k) \supset \mathcal{D}(\mathcal{A})$, $k = \overline{1,m}$,

$$\implies \quad (\mathcal{I}+\mathcal{S}_2)\mathcal{D}(\mathcal{A}_k) \supset (\mathcal{I}+\mathcal{S}_2)\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_{00}).$$

$\Longrightarrow$ problem (119) has a uniq. s.s. on $[0, T]$.
Inverse trans. $\Longrightarrow$ return from (119) to (109).
Return to $u(t)$ $\Longrightarrow$ $u(t)$ — s.s. of "associated" Cauchy problem (110).
Note:
In the proof we used assumption (113) for

$$(\mathcal{I} + \mathcal{S}_2)\mathcal{V}_k(t,\xi)(\mathcal{I} + \mathcal{S}_2)^{-1}, \quad \mathcal{V}_k(t,\xi) = \operatorname{diag}(V_k(t,\xi); 0), \quad k = \overline{1, m}.$$

□

Remark 3.2

If the cond. of Th. 3.6, 3.7 include additional smoothness cond. of s.s. (110) $\Longrightarrow$ $\exists$ a uniq. s.s. to Cauchy Problem (106), (107) for integro-diff. eq.

Remark 3.3

In (107) it was assumed that $G = 0$.
All results are generalized trivially for the case $G = G^* \in \mathcal{L}(\mathcal{H})$, and Th. 3.6, 3.7 hold true.

Remark 3.4

The case with general restrictions to oper. coeff. in eq. (106) may be studied by the scheme of S. 3.3.